Two Point Boundary-Value Problems

In the applications which we will study in this chapter, we will be confronted with the following problem.

PROBLEM: For which values of λ can we find nontrivial functions $y(x)$ which satisfy

$$\frac{d^2y}{dx^2} + \lambda y = 0; \qquad ay(0) + by'(0) = 0, \quad cy(l) + dy'(l) = 0? \tag{1}$$

Equation (1) is called a boundary-value problem, since we prescribe information about the solution $y(x)$ and its derivative $y'(x)$ at two distinct points, $x = 0$ and $x = l$. In an initial-value problem, on the other hand, we prescribe the value of y and its derivative at a single point $x = x_0$.

EXAMPLE: For which values of λ does the boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0; \qquad y(0) = 0, \quad y(l) = 0 \tag{2}$$

have nontrivial solutions?

Solution: We distinguish three cases.
(i) $\lambda = 0$. Every solution $y(x)$ of the differential equation $y'' = 0$ is of the form

$$y(x) = c_1x + c_2$$

for some choice of constants c_1 and c_2. The condition $y(0) = 0$ implies that $c_2 = 0$, and the condition $y(l) = 0$ then implies that $c_1 = 0$. Thus,

$$y(x) = 0$$

is the only solution of the boundary-value problem (2), for $\lambda = 0$.

(ii) $\lambda < 0$. In this case, every solution $y(x)$ of $y'' + \lambda y = 0$ is of the form

$$y(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$$

for some choice of constants c_1 and c_2. The boundary conditions

$$y(0) = 0, \quad y(l) = 0 \tag{2}$$

imply that

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{-\lambda}\,l} + c_2 e^{-\sqrt{-\lambda}\,l} = 0 \end{cases} \tag{3}$$

The system of equations (3) has a nonzero solution c_1, c_2 if, and only if,

$$\det \begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\,l} & e^{-\sqrt{-\lambda}\,l} \end{pmatrix} = e^{-\sqrt{-\lambda}\,l} - e^{\sqrt{-\lambda}\,l} = 0$$

This implies that

$$e^{\sqrt{-\lambda}\,l} = e^{-\sqrt{-\lambda}\,l} \qquad \Longrightarrow \qquad e^{2\sqrt{-\lambda}\,l} = 1$$

But this is impossible, since e^z is greater than one for $z > 0$. Hence, $c_1 = c_2 = 0$ and the boundary-value problem (2) has no nontrivial solutions $y(x)$ when λ is negative.

(iii) $\lambda > 0$. In this case, every solution $y(x)$ of $y'' + \lambda y = 0$ is of the form

$$y(x) = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x$$

for some choice of constants c_1 and c_2. The condition $y(0) = 0$ implies that $c_1 = 0$ and the condition $y(l) = 0$ then implies that

$$c_2 \sin \sqrt{\lambda}\, l = 0$$

This equation is satisfied, for any choice of c_2 if

$$\sqrt{\lambda}\, l = n\pi \quad \text{or} \quad \lambda = \frac{n^2\pi^2}{l^2}$$

for some positive integer n. Hence, the boundary-value problem (2) has nontrivial solutions

$$y(x) = c \sin\left(\frac{n\pi x}{l}\right) \quad \text{for} \quad \lambda = \frac{n^2\pi^2}{l^2},\ n = 1, 2, \ldots$$

REMARK: Our calculations for the case $\lambda < 0$ can be simplified if we write every solution

$$y(x) = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$$

in the form

$$y(x) = \tilde{c}_1 \cosh \sqrt{-\lambda}\, x + \tilde{c}_2 \sinh \sqrt{-\lambda}\, x$$

where

$$\cosh \sqrt{-\lambda}\, x = \frac{e^{\sqrt{-\lambda}\, x} + e^{-\sqrt{-\lambda}\, x}}{2}, \qquad \sinh \sqrt{-\lambda}\, x = \frac{e^{\sqrt{-\lambda}\, x} - e^{-\sqrt{-\lambda}\, x}}{2}$$

and

$$\tilde{c}_1 = c_1 + c_2, \qquad \tilde{c}_2 = c_1 - c_2 \qquad \text{(see Appendix)}$$

The condition $y(0) = 0$ implies that $\tilde{c}_1 = 0$, and the condition $y(l) = 0$ then implies that

$$\tilde{c}_2 \sinh \sqrt{-\lambda}\, l = 0$$

But $\sinh z$ is positive for $z > 0$. Hence, $\tilde{c}_2 = 0$ and $y(x) = 0$.

THEOREM: The boundary-value problem (1) has nontrivial solutions $y(x)$ only for a denumerable set of values

$$\lambda_1,\ \lambda_2, \ldots, \quad \text{where} \quad \lambda_1 \le \lambda_2 \ldots$$

and

$$\lambda_n \to \infty \quad \text{as} \quad n \to \infty$$

These special values of λ are called eigenvalues of (1), and the nontrivial solutions $y(x)$ are called eigenfunctions of (1). In this terminology, the eigenvalues of (2) are

$$\frac{\pi^2}{l^2},\quad \frac{4\pi^2}{l^2},\quad \frac{9\pi^2}{l^2}, \ldots$$

and the eigenfunctions of (2) are all constant multiples of

$$\sin\left(\frac{\pi x}{l}\right),\quad \sin\left(\frac{2\pi x}{l}\right), \ldots$$

EXAMPLE: Find the eigenvalues and eigenfunctions of the boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0; \qquad y(0) + y'(0) = 0, \quad y(1) = 0 \tag{4}$$

Solution: We distinguish three cases.
(i) $\lambda = 0$. Every solution $y(x)$ of the differential equation $y'' = 0$ is of the form

$$y(x) = c_1 x + c_2$$

for some choice of constants c_1 and c_2. The condition $y(0) + y'(0) = 0$ and $y(1) = 0$ both imply that $c_2 = -c_1$. Hence

$$y(x) = c(x-1), \ c \neq 0$$

is a nontrivial solution of (4) when $\lambda = 0$; i.e. $y(x) = c(x-1)$, $c \neq 0$, is an eigenfunction of (4) with eigenvalue zero.

(ii) $\lambda < 0$. In this case, every solution $y(x)$ of $y'' + \lambda y = 0$ is of the form

$$y(x) = c_1 \cosh \sqrt{-\lambda}\, x + c_2 \sinh \sqrt{-\lambda}\, x$$

for some choice of constants c_1 and c_2. The boundary conditions $y(0) + y'(0) = 0$ and $y(1) = 0$ imply that

$$\begin{cases} c_1 + \sqrt{-\lambda}\, c_2 = 0 \\ c_1 \cosh \sqrt{-\lambda} + c_2 \sinh \sqrt{-\lambda} = 0 \end{cases} \tag{5}$$

(Observe that $(\cosh x)' = \sinh x$ and $(\sinh x)' = \cosh x$.) The system of equations (5) has a nontrivial solution c_1, c_2 if, and only if,

$$\det \begin{pmatrix} 1 & \sqrt{-\lambda} \\ \cosh \sqrt{-\lambda} & \sinh \sqrt{-\lambda} \end{pmatrix} = \sinh \sqrt{-\lambda} - \sqrt{-\lambda} \cosh \sqrt{-\lambda} = 0$$

This implies that

$$\sinh \sqrt{-\lambda} = \sqrt{-\lambda} \cosh \sqrt{-\lambda} \tag{6}$$

But equation (6) has no solution $\lambda < 0$. To see this, let $z = \sqrt{-\lambda}$, and consider the function

$$h(z) = z \cosh z - \sinh z$$

This function is zero for $z = 0$ and is positive for $z > 0$, since its derivative

$$\begin{aligned} h'(z) &= (z \cosh z)' - (\sinh z)' \\ &= z' \cosh z + z(\cosh z)' - (\sinh z)' \\ &= \cosh z + z \sinh z - \cosh z \\ &= z \sinh z \end{aligned}$$

is strictly positive for $z > 0$. Hence, no negative number h can satisfy (6).

(iii) $\lambda > 0$. In this case, every solution $y(x)$ of $y'' + \lambda y = 0$ is of the form

$$y(x) = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x$$

for some choice of constants c_1 and c_2. The boundary conditions imply that

$$\begin{cases} c_1 + c_2\sqrt{\lambda} = 0 \\ c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} = 0 \end{cases} \tag{7}$$

The system of equations (7) has a nontrivial solution c_1, c_2 if, and only if,

$$\det\begin{pmatrix} 1 & \sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{pmatrix} = \sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda} = 0$$

This implies that

$$\tan\sqrt{\lambda} = \sqrt{\lambda} \tag{8}$$

To find those values of λ which satisfy (8), we set $\xi = \sqrt{\lambda}$ and draw the graphs of the functions $\mu = \xi$ and $\mu = \tan\xi$ in the $\xi - \mu$ plane (see Figure 1); the ξ coordinate of each point of intersection of these curves is then a root of the equation $\xi = \tan\xi$. It is clear that these curves intersect exactly once in the interval $\pi/2 < \xi < 3\pi/2$, and this occurs at a point $\xi_1 > \pi$. Similarly, these two curves intersect exactly once in the interval $3\pi/2 < \xi < 5\pi/2$, and this occurs at a point $\xi_2 > 2\pi$. More generally, the curves $\mu = \xi$ and $\mu = \tan\xi$ intersect exactly once in the interval

$$\frac{(2n-1)\pi}{2} < \xi < \frac{(2n+1)\pi}{2}$$

and this occurs at a point $\xi_n > n\pi$.

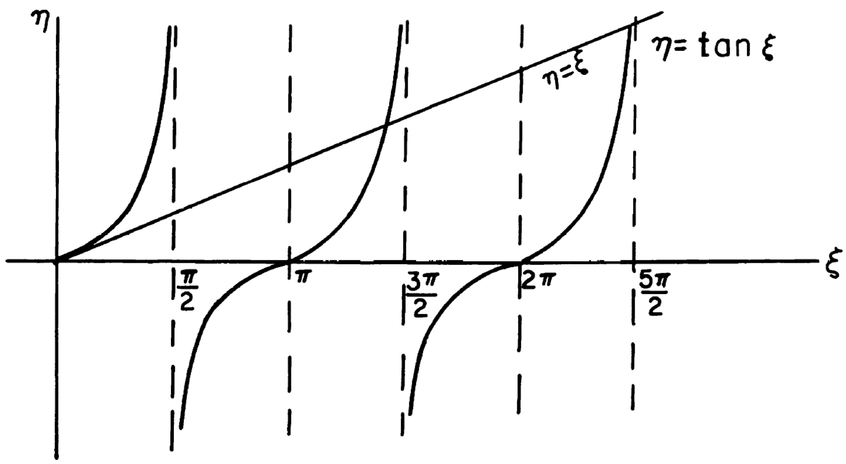

Figure 1. Graphs of $\eta = \xi$ and $\eta = \tan\xi$

Finally, the curves $\mu = \xi$ and $\mu = \tan\xi$ do not intersect in the interval $0 < \xi < \pi/2$. To prove this, set

$$h(\xi) = \tan\xi - \xi$$

and compute

$$h'(\xi) = \sec^2\xi - 1 = \tan^2\xi$$

This quantity is positive for $0 < \xi < \pi/2$. Consequently, the eigenvalues of (4) are $\lambda_1 = \xi_1^2,\ \lambda_2 = \xi_2^2, \ldots$, and the eigenfunction of (4) are all constant multiples of the functions

$$-\sqrt{\lambda_1}\cos\sqrt{\lambda_1}x + \sin\sqrt{\lambda_1}x, \quad -\sqrt{\lambda_2}\cos\sqrt{\lambda_2}x + \sin\sqrt{\lambda_2}x, \ldots$$

We cannot compute λ_n exactly. Nevertheless, we know that

$$n^2\pi^2 < \lambda_n < (2n+1)^2\frac{\pi^2}{4}$$

In addition, it is clear that λ_n approaches $(2n+1)^2\pi^2/4$ as n approaches infinity.

Appendix

Since

$$\cosh \sqrt{-\lambda}\, x = \frac{e^{\sqrt{-\lambda}\, x} + e^{-\sqrt{-\lambda}\, x}}{2}, \qquad \sinh \sqrt{-\lambda}\, x = \frac{e^{\sqrt{-\lambda}\, x} - e^{-\sqrt{-\lambda}\, x}}{2}$$

and

$$\tilde{c}_1 = c_1 + c_2, \qquad \tilde{c}_2 = c_1 - c_2$$

we have

$$\begin{aligned}
y(x) &= \tilde{c}_1 \cosh \sqrt{-\lambda}\, x + \tilde{c}_2 \sinh \sqrt{-\lambda}\, x \\
&= (c_1 + c_2) \frac{e^{\sqrt{-\lambda}\, x} + e^{-\sqrt{-\lambda}\, x}}{2} + (c_1 - c_2) \frac{e^{\sqrt{-\lambda}\, x} - e^{-\sqrt{-\lambda}\, x}}{2} \\
&= (c_1 + c_2) \left(\frac{1}{2} e^{\sqrt{-\lambda}\, x} + \frac{1}{2} e^{-\sqrt{-\lambda}\, x} \right) + (c_1 - c_2) \left(\frac{1}{2} e^{\sqrt{-\lambda}\, x} - \frac{1}{2} e^{-\sqrt{-\lambda}\, x} \right) \\
&= \frac{1}{2}(c_1 + c_2) e^{\sqrt{-\lambda}\, x} + \frac{1}{2}(c_1 + c_2) e^{-\sqrt{-\lambda}\, x} + \frac{1}{2}(c_1 - c_2) e^{\sqrt{-\lambda}\, x} - \frac{1}{2}(c_1 - c_2) e^{-\sqrt{-\lambda}\, x} \\
&= \frac{1}{2}(c_1 + c_2 + c_1 - c_2) e^{\sqrt{-\lambda}\, x} + \frac{1}{2}(c_1 + c_2 - c_1 + c_2) e^{-\sqrt{-\lambda}\, x} \\
&= \frac{1}{2}(2c_1) e^{\sqrt{-\lambda}\, x} + \frac{1}{2}(2c_2) e^{-\sqrt{-\lambda}\, x} \\
&= c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}
\end{aligned}$$