Phase Portraits of Linear Systems

In this section we present a complete picture of all orbits of the linear differential equation

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

This picture is called a *phase portrait*, and it depends almost completely on the eigenvalues of the matrix A. It also changes drastically as the eigenvalues of A change sign or become imaginary.

When analyzing equation (1), it is often helpful to visualize a vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in $\mathbb{R}^2$ as a direction, or directed line segment, in the plane. Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be a vector in $\mathbb{R}^2$ and draw the directed line segment $\vec{x}$ from the point $(0,0)$ to the point (x_1, x_2), as in Figure 1a. This directed line segment is parallel to the line through $(0,0)$ with direction numbers x_1, x_2 respectively. If we visualize the vector $\mathbf{x}$ as being this directed line segment $\vec{x}$, then we see that the vectors $\mathbf{x}$ and $c\mathbf{x}$ are parallel if c is positive, and antiparallel if c is negative. We can also give a nice geometric interpretation of vector addition. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^2$. Draw the directed line segment $\vec{x}$, and place the vector $\vec{y}$ at the tip of $\vec{x}$. The vector $\vec{x} + \vec{y}$ is then the composition of these two directed line segments (see Figure 2). This construction is known as the parallelogram law of vector addition.

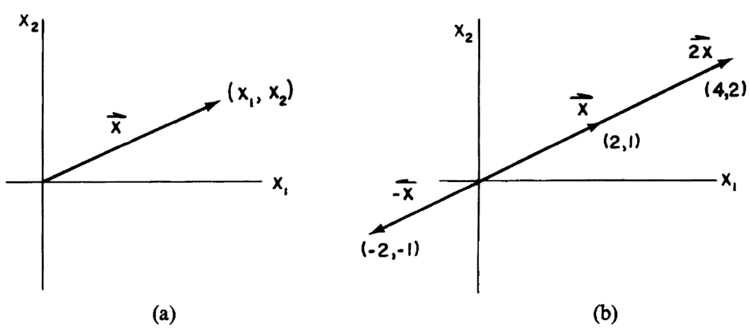

Figure 1

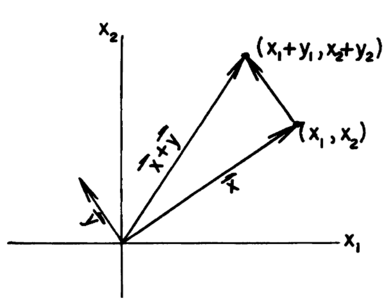

Figure 2

We are now in a position to derive the phase portraits of (1). Let λ_1 and λ_2 denote the two eigenvalues of A. We distinguish the following cases.

1. $\lambda_2 < \lambda_1 < 0$. Let $\mathbf{v^1}$ and $\mathbf{v^2}$ be eigenvectors of A with eigenvalues λ_1 and λ_2 respectively. In the $x_1 - x_2$ plane we draw the four half-lines l_1, l_1', l_2, and l_2', as shown in Figure 3. The rays l_1 and l_2 are parallel to $\mathbf{v^1}$ and $\mathbf{v^2}$, while the rays l_1' and l_2' are parallel to $-\mathbf{v^1}$ and $-\mathbf{v^2}$. In this case, the phase portrait of (1) has the form described in Figure 3.

The distinguishing feature of this phase portrait is that every orbit, with the exception of a single line, approaches the origin in a fixed direction (if we consider the directions $\mathbf{v^1}$ and $-\mathbf{v^1}$ equivalent). In this case we say that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ of (1) is a *stable node.*

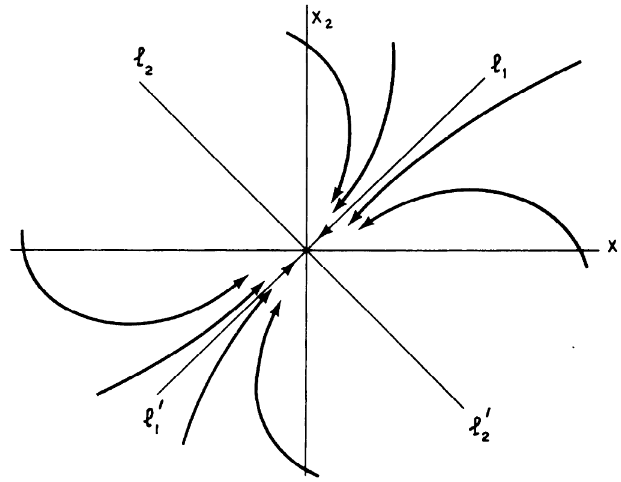

Figure 3. Phase portrait of a stable node

REMARK: The orbit of every solution $\mathbf{x}(t)$ of (1) approaches the origin $x_1 = x_2 = 0$ as t approaches infinity. However, this point does not belong to the orbit of any nontrivial solution $\mathbf{x}(t)$.

$1'$. $0 < \lambda_1 < \lambda_2$. The phase portrait of (1) in this case is exactly the same as Figure 3, except that the direction of the arrows is reversed. Hence, the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ of (1) is an *unstable node* if both eigenvalues of A are positive.

EXAMPLE: Draw the phase portrait of the linear equation

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} -2 & -1 \\ 4 & -7 \end{bmatrix} \mathbf{x} \tag{2}$$

Solution: It is easily verified that

$$\mathbf{v^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v^2} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

are eigenvectors of A with eigenvalues -3 and -6, respectively, and $\mathbf{x} = \begin{bmatrix} c_1 e^{-3t} + c_2 e^{-6t} \\ c_1 e^{-3t} + 4c_2 e^{-6t} \end{bmatrix}$ (see Appendix I). Therefore, $\mathbf{x} = \mathbf{0}$ is a stable node of (2), and the phase portrait of (2) has the form described in Figure 7. The half-line l_1 makes an angle of 45° with the x_1-axis, while the half-line l_2 makes an angle of θ degrees with the x_1-axis, where $\tan\theta = 4$.

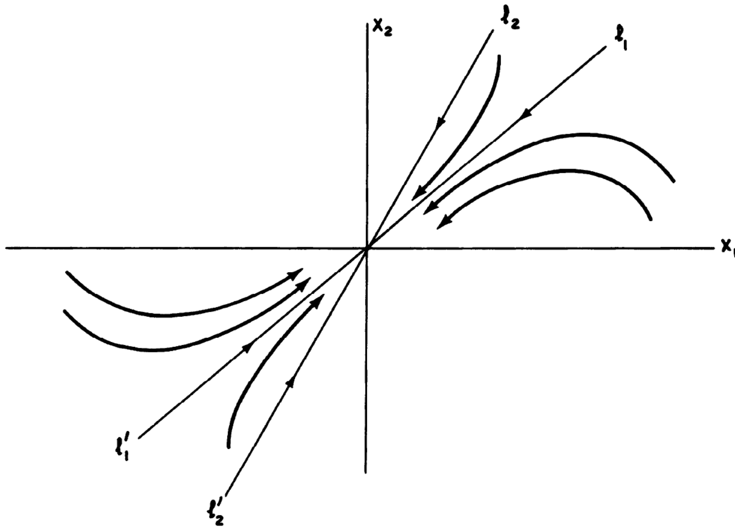

Figure 7. Phase portrait of (8)

2. $\lambda_1 = \lambda_2 < 0$. In this case, the phase portrait of (1) depends on whether A has one or two linearly independent eigenvectors.

(a) Suppose that A has two linearly independent eigenvectors $\mathbf{v^1}$ and $\mathbf{v^2}$ with eigenvalue $\lambda < 0$. In this case, the phase portrait of (1) has the form described in Figure 4a. That is, the orbit of every solution $\mathbf{x}(t)$ of (1) is a half-line. Moreover, the set of vectors $\{c_1\mathbf{v^1} + c_2\mathbf{v^2}\}$, for all choices of c_1 and c_2 cover every direction in the $x_1 - x_2$ plane, since $\mathbf{v^1}$ and $\mathbf{v^2}$ are linearly independent.

(b) Suppose that A has only one linearly independent eigenvector $\mathbf{v}$, with eigenvalue λ. In this case, the phase portrait of (1) has the form described in Figure 4b. That is, every solution $\mathbf{x}(t)$ of (1) approaches $\binom{0}{0}$ as t approaches infinity. Moreover, the tangent to the orbit of $\mathbf{x}(t)$ approaches $\pm\mathbf{v}$ (depending on the sign of c_2) as t approaches infinity.

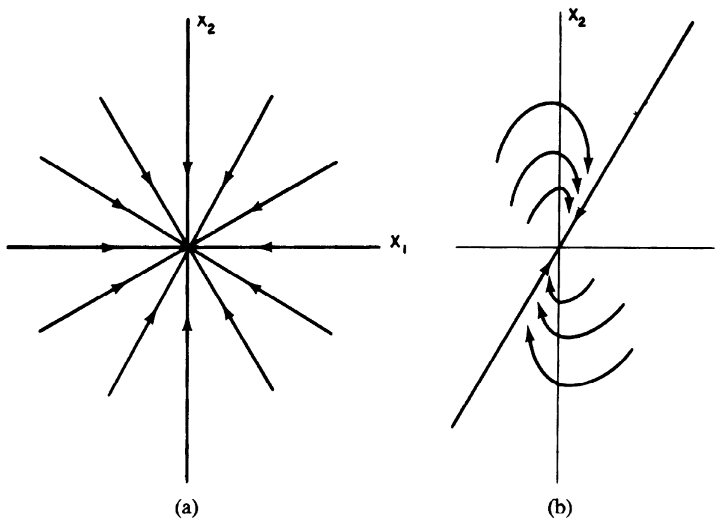

Figure 4

2$'$. $\lambda_1 = \lambda_2 > 0$. The phase portraits of (1) in the cases $(2a)'$ and $(2b)'$ are exactly the same as Figures 4a and 4b, except that the direction of the arrows is reversed.

3. $\lambda_1 < 0 < \lambda_2$. Let $\mathbf{v^1}$ and $\mathbf{v^2}$ be eigenvectors of A with eigenvalues λ_1 and λ_2 respectively. In the $x_1 - x_2$ plane we draw the four half-lines l_1, l_1', l_2, and l_2'; the half-lines l_1 and l_2 are parallel to $\mathbf{v^1}$ and $\mathbf{v^2}$, while the half-lines l_1' and l_2' are parallel to $-\mathbf{v^1}$ and $-\mathbf{v^2}$. In this case, the phase portrait of (1) has the form described in Figure 5. This phase portrait resembles a "saddle" near $x_1 = x_2 = 0$. For this reason, we say that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ of (1) is a *saddle point* if the eigenvalues of A have opposite sign.

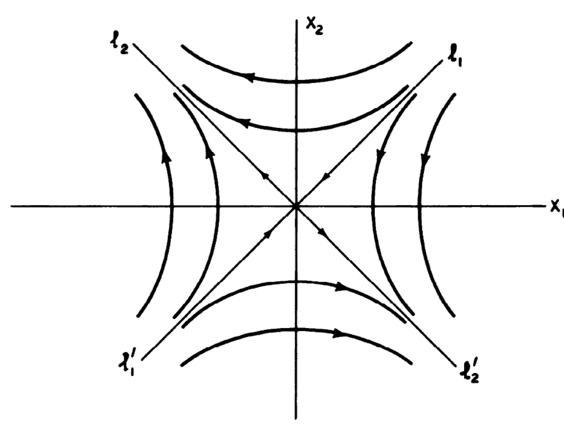

Figure 5. Phase portrait of a saddle point

EXAMPLE: Draw the phase portrait of the linear equation

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \mathbf{x} \tag{3}$$

Solution: It is easily verified that $\mathbf{v^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v^2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of A with eigenvalues -2

and 4, respectively and

$$\mathbf{x} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{4t} \\ c_1 e^{-2t} - c_2 e^{4t} \end{bmatrix}$$

Therefore, $\mathbf{x} = \mathbf{0}$ is a saddle point of (3), and its phase portrait has the form described in Figure 8. The half-line l_1 makes an angle of 45° with the x_1-axis, and the half-line l_2 is at right angles to l_1.

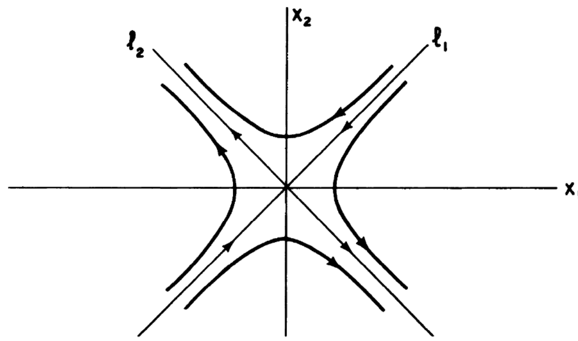

Figure 8. Phase portrait of (9)

4. $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta, \beta \neq 0$. We distinguish the following cases.

(a) $\alpha = 0$. In this case, the phase portrait of (1) has the form described in Figure 6a. For this reason, we say that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ of (1) is a *center* when the eigenvalues of A are pure imaginary.

The direction of the arrows in Figure 6a must be determined from the differential equation (1). The simplest way of doing this is to check the sign of $\dot{x}_2$ when $x_2 = 0$. If $\dot{x}_2$ greater than zero for $x_2 = 0$ and $x_1 > 0$ (that is, if c in the matrix A is > 0), then all solutions $\mathbf{x}(t)$ of (1) move in the counterclockwise direction; if $\dot{x}_2$ is less than zero for $x_2 = 0$ and $x_1 > 0$ (that is, if c in A is < 0), then all solutions $\mathbf{x}(t)$ of (1) move in the clockwise direction.

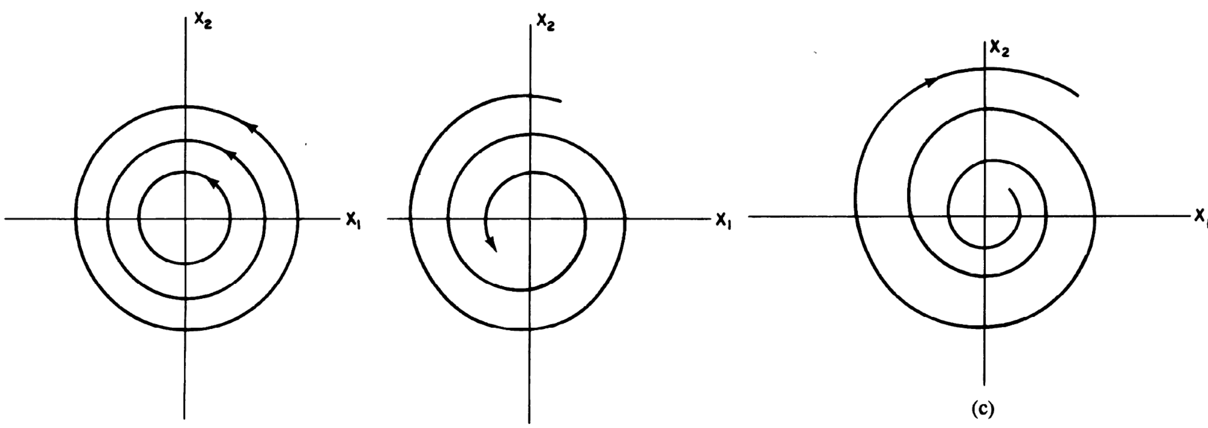

Figure 6. (a) $\alpha = 0$; (b) $\alpha < 0$; (c) $\alpha > 0$

(b) $\alpha < 0$. In this case, the phase portrait of (1) has the form described in Figure 6b, and we say that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ of (1) is a *stable focus.* The direction of rotation of the spiral in Figure 6b must be determined directly from the differential equation (1). That is, if c in A is > 0, then all nontrivial orbits of (1) spiral into the origin in the counterclockwise direction. Otherwise, all nontrivial orbits of (1) spiral into the origin in the clockwise direction.

(c) $\alpha > 0$. In this case, all orbits of (1) spiral away from the origin as t approaches infinity (see Figure 6c), and the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ of (1) is called an *unstable focus.* Again, the direction of rotation of the spiral in Figure 6c must be determined directly from the differential equation (1).

THEOREM 4: Suppose that $\mathbf{u} = \mathbf{0}$ is either a node, saddle, or focus point of the differential equation $\dot{\mathbf{u}} = A\mathbf{u}$. Then, the phase portrait of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, in a neighborhood of $\mathbf{x} = \mathbf{x}^{\mathbf{0}}$, has one of the forms described in Figures 3, 5, and 6 (b and c), depending as to whether $\mathbf{u} = \mathbf{0}$ is a node, saddle, or focus.

EXAMPLE: Draw the phase portrait of the linear equation

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x} \tag{4}$$

Solution: The eigenvalues of A are $-1 \pm i$ and

$$\mathbf{x} = \begin{bmatrix} e^{-t}(c_1 \sin t + c_2 \cos t) \\ e^{-t}(c_1 \cos t - c_2 \sin t) \end{bmatrix}$$

Since $\alpha = -1 < 0$, the equilibrium solution $\mathbf{x} = \mathbf{0}$ is a stable focus of (4) and every nontrivial orbit of (4) spirals into the origin as t approaches infinity. To determine the direction of rotation of the spiral, we observe that $\dot{x}_2 = -x_1$ when $x_2 = 0$. Thus, $\dot{x}_2$ negative for $x_1 > 0$ and $x_2 = 0$. Consequently, all nontrivial orbits of (4) spiral into the origin in the clockwise direction, as shown in Figure 9. In short, since $c = -1 < 0$ in A, all nontrivial orbits of (4) spiral into the origin in the clockwise direction.

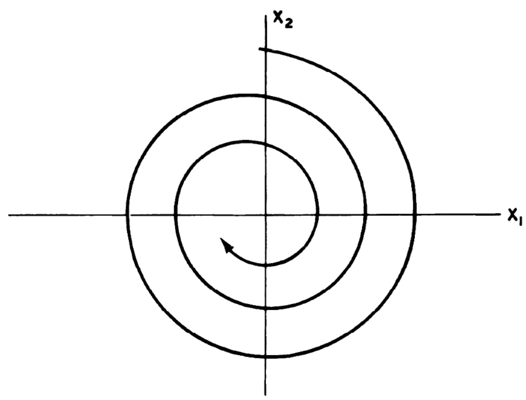

Figure 9. Phase portrait of (10)

Appendix I

Note that system (2) can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -2 & -1 \\ 4 & -7 \end{bmatrix}$$

The characteristic polynomial of the matrix A is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ 4 & -7-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-7-\lambda) - (-1)(4) = 14 + 2\lambda + 7\lambda + \lambda^2 + 4 = \lambda^2 + 9\lambda + 18$$

so the eigenvalues of A are $\lambda = -3$ and $\lambda = -6$.

(a) Let $\lambda = -3$. We use row operations:

$$\left[\begin{array}{cc|c} -2-\lambda & -1 & 0 \\ 4 & -7-\lambda & 0 \end{array}\right] = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -4 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - x_2 = 0 \qquad \Longrightarrow \qquad x_1 = x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = -3$. Consequently,

$$ce^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^1}(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Let $\lambda = -6$. We use row operations:

$$\begin{bmatrix} -2-\lambda & -1 & 0 \\ 4 & -7-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ 4 & -1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 4 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - \frac{1}{4}x_2 = 0 \qquad \Longrightarrow \qquad x_1 = \frac{1}{4}x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}x_2 \\ x_2 \end{bmatrix} = \frac{1}{4}x_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = -6$. Consequently,

$$ce^{-6t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^1}(t) = e^{-6t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The solutions $\mathbf{x^1}(t)$ and $\mathbf{x^2}(t)$ must be linearly independent, since A has distinct eigenvalues. Therefore, every solution $\mathbf{x}(t)$ must be of the form

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 e^{-3t} + c_2 e^{-6t} \\ c_1 e^{-3t} + 4c_2 e^{-6t} \end{bmatrix}$$