

The Phase-Plane

In this section we begin our study of the “geometric” theory of differential equations. For simplicity, we will restrict ourselves, for the most part, to the case $n = 2$. Our aim is to obtain as complete a description as possible of all solutions of the system of differential equations

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y) \quad (1)$$

To this end, observe that every solution $x = x(t), y = y(t)$ of (1) defines a curve in the three-dimensional space t, x, y . That is to say, the set of all points $(t, x(t), y(t))$ describe a curve in the three-dimensional space t, x, y . For example, the solution $x = \cos t, y = \sin t$ (see Appendix I) of the system of differential equations

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

describes a helix (see Figure 1) in (t, x, y) space.

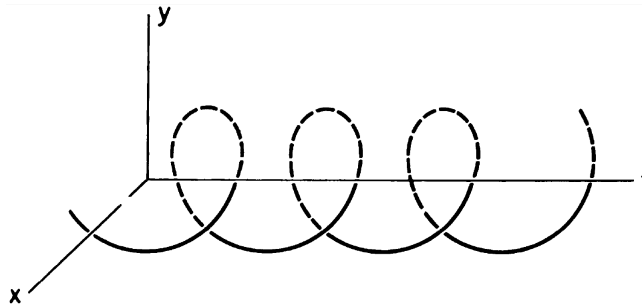


Figure 1. Graph of the solution $x = \cos t, y = \sin t$

The geometric theory of differential equations begins with the important observation that every solution $x = x(t), y = y(t), t_0 \leq t \leq t_1$, of (1) also defines a curve in the $x - y$ plane. To wit, as t runs from t_0 to t_1 , the set of points $(x(t), y(t))$ trace out a curve C in the $x - y$ plane. This curve is called the *orbit*, or *trajectory*, of the solution $x = x(t), y = y(t)$, and the $x - y$ plane is called the *phase-plane* of the solutions of (1). Equivalently, we can think of the orbit of $x(t), y(t)$ as the path that the solution traverses in the $x - y$ plane.

EXAMPLE: It is easily verified that

$$x = \cos t, \quad y = \sin t$$

is a solution of the system of differential equations $\dot{x} = -y, \dot{y} = x$. As t runs from 0 to 2π , the set of points $(\cos t, \sin t)$ trace out the unit circle $x^2 + y^2 = 1$ in the $x - y$ plane. Hence, the unit circle $x^2 + y^2 = 1$ is the orbit of the solution $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$. As t runs from 0 to ∞ , the set of points $(\cos t, \sin t)$ trace out this circle infinitely often.

EXAMPLE: It is easily verified that (see Appendix II)

$$x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad -\infty < t < \infty$$

is a solution of the system of differential equations

$$\frac{dx}{dt} = -x - y, \quad \frac{dy}{dt} = x - y$$

As t runs from $-\infty$ to ∞ , the set of points $(e^{-t} \cos t, e^{-t} \sin t)$ trace out a spiral in the $x - y$ plane. Hence, the orbit of the solution $x = e^{-t} \cos t, y = e^{-t} \sin t$ is the spiral shown in Figure 2.

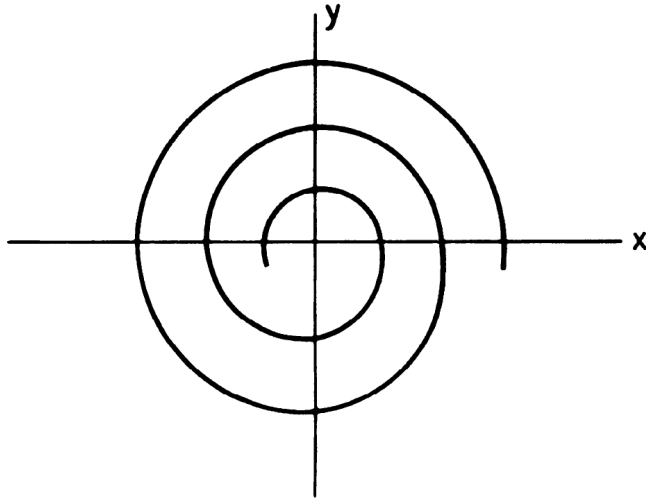


Figure 2. Orbit of $x = e^{-t} \cos t$, $y = e^{-t} \sin t$

One of the advantages of considering the orbit of the solution rather than the solution itself is that it is often possible to obtain the orbit of a solution without prior knowledge of the solution. Let $x = x(t)$, $y = y(t)$ be a solution of (1). If $x'(t)$ is unequal to zero at $t = t_1$, then we can solve for $t = t(x)$ in a neighborhood of the point $x_1 = x(t_1)$. Thus, for t near t_1 , the orbit of the solution $x(t)$, $y(t)$ is the curve $y = y(t(x))$. Next, observe that

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x, y)}{f(x, y)}$$

Thus, the orbits of the solutions $x = x(t)$, $y = y(t)$ of (1) are the solution curves of the first-order scalar equation

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \quad (2)$$

Therefore, it is not necessary to find a solution $x(t)$, $y(t)$ of (1) in order to compute its orbit; we need only solve the single first-order scalar differential equation (2).

REMARK: From now on, we will use the phrase “the orbits of (1)” to denote the totality of orbits of solutions of (1).

EXAMPLE: The orbits of the system of differential equations

$$\frac{dx}{dt} = y^2, \quad \frac{dy}{dt} = x^2 \quad (3)$$

are the solution curves of the scalar equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x^2}{y^2}$$

This equation is separable, and it is easily seen that every solution is of the form

$$y(x) = (x^3 - c)^{1/3}, \quad c \text{ constant}$$

Thus, the orbits of (3) are the set of all curves $y(x) = (x^3 - c)^{1/3}$.

EXAMPLE: The orbits of the system of differential equations

$$\frac{dx}{dt} = y(1 + x^2 + y^2), \quad \frac{dy}{dt} = -2x(1 + x^2 + y^2) \quad (4)$$

are the solution curves of the scalar equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{2x(1+x^2+y^2)}{y(1+x^2+y^2)} = -\frac{2x}{y}$$

This equation is separable, and all solutions are of the form

$$\frac{1}{2}y^2 + x^2 = c^2$$

Hence, the orbits of (4) are the families of ellipses $\frac{1}{2}y^2 + x^2 = c^2$.

WARNING: A solution curve of (2) is an orbit of (1) only if dx/dt and dy/dt are not zero simultaneously along the solution. If a solution curve of (2) passes through an equilibrium point of (1), then the entire solution curve is not an orbit. Rather, it is the union of several distinct orbits. For example, consider the system of differential equations

$$\frac{dx}{dt} = y(1-x^2-y^2), \quad \frac{dy}{dt} = -x(1-x^2-y^2) \quad (5)$$

The solution curves of the scalar equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x(1-x^2-y^2)}{y(1-x^2-y^2)} = -\frac{x}{y}$$

are the family of concentric circles $x^2 + y^2 = c^2$. Observe, however, that every point on the unit circle $x^2 + y^2 = 1$ is an equilibrium point of (5). Thus, the orbits of this system are the circles $x^2 + y^2 = c^2$, for $c \neq 1$, and all points on the unit circle $x^2 + y^2 = 1$. Similarly, the orbits of (3) are the curves $y = (x^3 - c)^{1/3}$, $c \neq 0$; the half-lines $y = x$, $x > 0$, and $y = x$, $x < 0$; and the point $(0, 0)$.

Appendix I

Here we solve the system

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = x \end{cases}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and its characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (-\lambda)^2 - (1)(-1) = \lambda^2 + 1$$

Hence the eigenvalues of A are $\lambda = \pm i$. Let $\lambda = i$. We use row operations:

$$\begin{bmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/i & 0 \\ 1 & -i & 0 \end{bmatrix}$$

Since

$$\frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{-1} = -i$$

it follows that

$$\begin{bmatrix} 1 & 1/i & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -i & 0 \\ 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 - ix_2 = 0 \quad \implies \quad x_1 = -ix_2 \quad \implies \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

is the eigenvector of A , corresponding to $\lambda = i$. Consequently,

$$\mathbf{x}(t) = e^{it} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

is a complex-valued solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Now,

$$\begin{aligned} e^{it} \begin{bmatrix} -i \\ 1 \end{bmatrix} &= (\cos t + i \sin t) \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -i \cos t + \sin t \\ \cos t + i \sin t \end{bmatrix} \\ &= \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} -i \cos t \\ i \sin t \end{bmatrix} \\ &= \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} \end{aligned}$$

Therefore, by Lemma 1 from Section 3.9,

$$\mathbf{x}^1(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}^2(t) = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

are real-valued solutions of the system. Therefore the general solution is

$$\mathbf{x}(t) = c_1 \mathbf{x}^1(t) + c_2 \mathbf{x}^2(t) = c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

Appendix II

Here we solve the system

$$\begin{cases} \frac{dx}{dt} = -x - y \\ \frac{dy}{dt} = x - y \end{cases}$$

The matrix of the system is

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

and its characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1 - \lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} = (-1 - \lambda)^2 - (1)(-1) = \lambda^2 + 2\lambda + 2$$

Hence the eigenvalues of A are

$$\begin{aligned} \lambda &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{4(-1)}}{2} = \frac{-2 \pm \sqrt{4}\sqrt{-1}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= \frac{-2}{2} \pm \frac{2i}{2} \\ &= -1 \pm i \end{aligned}$$

Let $\lambda = -1 + i$. We use row operations:

$$\begin{bmatrix} -1 - \lambda & -1 & 0 \\ 1 & -1 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/i & 0 \\ 1 & -i & 0 \end{bmatrix}$$

Since

$$\frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{-1} = -i$$

it follows that

$$\begin{bmatrix} 1 & 1/i & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -i & 0 \\ 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 - ix_2 = 0 \quad \implies \quad x_1 = -ix_2 \quad \implies \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

is the eigenvector of A , corresponding to $\lambda = -1 + i$. Consequently,

$$\mathbf{x}(t) = e^{(-1+i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

is a complex-valued solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Now,

$$\begin{aligned} e^{(-1+i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} &= e^{-t+it} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-t} e^{it} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-t} (\cos t + i \sin t) \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} -i \cos t + \sin t \\ \cos t + i \sin t \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + e^{-t} \begin{bmatrix} -i \cos t \\ i \sin t \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + i e^{-t} \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} \end{aligned}$$

Therefore, by Lemma 1 from Section 3.9,

$$\mathbf{x}^1(t) = e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}^2(t) = e^{-t} \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

are real-valued solutions of the system. Therefore the general solution is

$$\mathbf{x}(t) = c_1 \mathbf{x}^1(t) + c_2 \mathbf{x}^2(t) = c_1 e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$