Stability of Linear Systems

In this section we consider the stability question for solutions of the **autonomous** differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

Let

$$\mathbf{x} = \boldsymbol{\phi}(t) = \begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_n(t) \end{bmatrix}$$

be a solution of (1). We are interested in determining whether $\boldsymbol{\phi}(t)$ is stable or unstable. That is to say, we seek to determine whether every solution

$$\boldsymbol{\psi}(t) = \begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix}$$

of (1) which starts sufficiently close to $\boldsymbol{\phi}(t)$ at $t = 0$ must remain close to $\boldsymbol{\phi}(t)$ for all future time $t \geq 0$. We begin with the following formal definition of stability.

DEFINITION: The solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of (1) is **stable** if every solution $\boldsymbol{\psi}(t)$ of (1) which starts sufficiently close to $\boldsymbol{\phi}(t)$ at $t = 0$ must remain close to $\boldsymbol{\phi}(t)$ for all future time t. The solution $\boldsymbol{\phi}(t)$ is **unstable** if there exists at least one solution $\boldsymbol{\psi}(t)$ of (1) which starts near $\boldsymbol{\phi}(t)$ at $t = 0$ but which does not remain close to $\boldsymbol{\phi}(t)$ for all future time. More precisely, the solution $\boldsymbol{\phi}(t)$ is stable if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that

$$|\boldsymbol{\psi}_j(t) - \boldsymbol{\phi}_j(t)| < \epsilon \quad \text{if} \quad |\boldsymbol{\psi}_j(0) - \boldsymbol{\phi}_j(0)| < \delta, \qquad j = 1, \ldots, n$$

for every solution $\boldsymbol{\psi}(t)$ of (1).

The stability question can be completely resolved for each solution of the linear differential equation

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{2}$$

THEOREM 1:

(a) Every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of (2) is stable if all the eigenvalues of A have negative real part.

(b) Every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of (2) is unstable if at least one eigenvalue of A has positive real part.

(c) Suppose that all the eigenvalues of A have real part ≤ 0 and $\lambda_1 = i\sigma_1, \ldots, \lambda_l = i\sigma_l$ have zero real part. Let $\lambda_j = i\sigma_j$ have multiplicity k_j. This means that the characteristic polynomial of A can be factored into the form

$$p(\lambda) = (\lambda - i\sigma_1)^{k_1} \ldots (\lambda - i\sigma_l)^{k_l} q(\lambda)$$

where all the roots of $q(\lambda)$ have negative real part. Then, every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of (2) is stable if A has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j = i\sigma_j$. Otherwise, every solution $\boldsymbol{\phi}(t)$ is unstable.

DEFINITION: A solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of (1) is **asymptotically stable** if it is stable, and if every solution $\boldsymbol{\psi}(t)$ which starts sufficiently close to $\boldsymbol{\phi}(t)$ must approach $\boldsymbol{\phi}(t)$ as t approaches infinity. In particular, an equilibrium solution $\mathbf{x}(t) = \mathbf{x^0}$ of (1) is asymptotically stable if every solution $\mathbf{x} = \boldsymbol{\psi}(t)$ of (1) which starts sufficiently close to $\mathbf{x^0}$ at time $t = 0$ not only remains close to $\mathbf{x^0}$ for all future time, but ultimately approaches $\mathbf{x^0}$ as t approaches infinity.

REMARK: The asymptotic stability of any solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of (2) is clearly equivalent to the asymptotic stability of the equilibrium solution $\mathbf{x}(t) \equiv \mathbf{0}$.

EXAMPLE: Determine whether each solution $\mathbf{x}(t)$ of the differential equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{bmatrix} \mathbf{x}$$

is stable, asymptotically stable, or unstable.

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{bmatrix}$$

is

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 2 \\ -3 & -2 & -1-\lambda \end{vmatrix} &= (-1-\lambda) \begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} \\ &= (-1-\lambda)[(-1-\lambda)^2 + 4] \\ &= (-1-\lambda)(1 + 2\lambda + \lambda^2 + 4) \\ &= -(1+\lambda)(\lambda^2 + 2\lambda + 5) \end{aligned}$$

Hence, $\lambda = -1$ and

$$\begin{aligned} \lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{4^2(-1)}}{2} &= \frac{-2 \pm \sqrt{4^2}\sqrt{-1}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

are the eigenvalues of A. Since all three eigenvalues have negative real part, we conclude that every solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable.

EXAMPLE: Prove that every solution of the differential equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \mathbf{x}$$

is unstable.

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 25$$

Hence, $\lambda = 6$ and $\lambda = -4$ are the eigenvalues of A. Since one eigenvalue of A is positive, we conclude that every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ is unstable.

EXAMPLE: Show that every solution of the differential equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

is stable, but not asymptotically stable.

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$$

is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 6$$

Hence, $\lambda = \pm\sqrt{6}i$ are the eigenvalues of A. Therefore, by part (c) of Theorem 1, every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ is stable. However, no solution is asymptotically stable. This follows immediately from the fact that the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is (see the Appendix)

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -\sqrt{6}\sin(\sqrt{6}t) \\ 2\cos(\sqrt{6}t) \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{6}\cos(\sqrt{6}t) \\ 2\sin(\sqrt{6}t) \end{bmatrix}$$

Hence, every solution $\mathbf{x}(t)$ is periodic, with period $2\pi/\sqrt{6}$, and no solution $\mathbf{x}(t)$ (except $\mathbf{x}(t) \equiv \mathbf{0}$) approaches 0 as t approaches infinity.

EXAMPLE: Show that every solution of the differential equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{bmatrix} \mathbf{x}$$

is unstable.

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{bmatrix}$$

is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 0 \\ 0 & -6-\lambda & -2 \\ -6 & 0 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -6-\lambda & -2 \\ 0 & -3-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 0 & -2 \\ -6 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-6-\lambda)(-3-\lambda) - (-3)(-12)$$

$$= (-12 - 2\lambda + 6\lambda + \lambda^2)(-3-\lambda) - 36$$

$$= (-12 + 4\lambda + \lambda^2)(-3-\lambda) - 36$$

$$= 36 - 12\lambda - 3\lambda^2 + 12\lambda - 4\lambda^2 - \lambda^3 - 36$$

$$= -7\lambda^2 - \lambda^3$$

$$= -\lambda^2(7+\lambda)$$

so the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = -7$. Let $\lambda = 0$. We use row operations:

$$\begin{bmatrix} 2-\lambda & -3 & 0 & 0 \\ 0 & -6-\lambda & -2 & 0 \\ -6 & 0 & -3-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ -6 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & -9 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 1 & 1/3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_1 + \dfrac{1}{2}x_3 = 0 \\ \\ x_2 + \dfrac{1}{3}x_3 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = -\dfrac{1}{2}x_3 \\ \\ x_2 = -\dfrac{1}{3}x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2}x_3 \\ -\dfrac{1}{3}x_3 \\ x_3 \end{bmatrix} = -\frac{1}{6}x_3 \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 0$. Consequently, every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ is unstable, since $\lambda = 0$ is an eigenvalue of multiplicity two and A has only one linearly independent eigenvector with eigenvalue 0.

Appendix

EXAMPLE: Find all solutions of the equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

Solution: The characteristic polynomial of the matrix $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$ is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 6$$

Hence, $\lambda = \pm\sqrt{6}i$ are the eigenvalues of A. Let $\lambda = \sqrt{6}i$. We use row operations:

$$\begin{bmatrix} -\lambda & -3 & 0 \\ 2 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{6}i & -3 & 0 \\ 2 & -\sqrt{6}i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \dfrac{3}{\sqrt{6}i} & 0 \\ 1 & -\dfrac{\sqrt{6}}{2}i & 0 \end{bmatrix}$$

Since

$$\frac{3}{\sqrt{6}i} = \frac{3 \cdot \sqrt{6}i}{\sqrt{6}i \cdot \sqrt{6}i} = \frac{3\sqrt{6}i}{6(-1)} = -\frac{\sqrt{6}}{2}i$$

it follows that

$$\begin{bmatrix} 1 & \dfrac{3}{\sqrt{6}i} & 0 \\ 1 & -\dfrac{\sqrt{6}}{2}i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\dfrac{\sqrt{6}}{2}i & 0 \\ 1 & -\dfrac{\sqrt{6}}{2}i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\dfrac{\sqrt{6}}{2}i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 - \frac{\sqrt{6}}{2}ix_2 = 0 \qquad \Longrightarrow \qquad x_1 = \frac{\sqrt{6}}{2}ix_2 \qquad \Longrightarrow \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{6}}{2}ix_2 \\ x_2 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} \sqrt{6}i \\ 2 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = \sqrt{6}i$. Consequently, the complex-valued solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\begin{aligned}
\mathbf{x}(t) = e^{\sqrt{6}it} \begin{bmatrix} \sqrt{6}i \\ 2 \end{bmatrix} &= (\cos(\sqrt{6}t) + i\sin(\sqrt{6}t)) \begin{bmatrix} 0 + \sqrt{6}i \\ 2 + 0i \end{bmatrix} \\
&= (\cos(\sqrt{6}t) + i\sin(\sqrt{6}t)) \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} \sqrt{6} \\ 0 \end{bmatrix} \right) \\
&= \cos(\sqrt{6}t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} + i\cos(\sqrt{6}t) \begin{bmatrix} \sqrt{6} \\ 0 \end{bmatrix} + i\sin(\sqrt{6}t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \sin(\sqrt{6}t) \begin{bmatrix} \sqrt{6} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 2\cos(\sqrt{6}t) \end{bmatrix} + i \begin{bmatrix} \sqrt{6}\cos(\sqrt{6}t) \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2\sin(\sqrt{6}t) \end{bmatrix} - \begin{bmatrix} \sqrt{6}\sin(\sqrt{6}t) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\sqrt{6}\sin(\sqrt{6}t) \\ 2\cos(\sqrt{6}t) \end{bmatrix} + i \begin{bmatrix} \sqrt{6}\cos(\sqrt{6}t) \\ 2\sin(\sqrt{6}t) \end{bmatrix}
\end{aligned}$$

Consequently, by Lemma 1,

$$\mathbf{x^1}(t) = \begin{bmatrix} -\sqrt{6}\sin(\sqrt{6}t) \\ 2\cos(\sqrt{6}t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{x^2}(t) = \begin{bmatrix} \sqrt{6}\cos(\sqrt{6}t) \\ 2\sin(\sqrt{6}t) \end{bmatrix}$$

are linearly independent real-valued solutions. Therefore, the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -\sqrt{6}\sin(\sqrt{6}t) \\ 2\cos(\sqrt{6}t) \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{6}\cos(\sqrt{6}t) \\ 2\sin(\sqrt{6}t) \end{bmatrix}$$