Introduction

In this chapter we consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix}$$

is a nonlinear function of $x_1, \ldots, x_n$. Unfortunately, there are no known methods of solving equation (1). This, of course, is very disappointing. However, it is not necessary, in most applications, to find the solutions of (1) explicitly. For example, let $x_1(t)$ and $x_2(t)$ denote the populations, at time t, of two species competing amongst themselves for the limited food and living space in their microcosm. Suppose, moreover, that the rates of growth of $x_1(t)$ and $x_2(t)$ are governed by the differential equation (1). In this case, we are not really interested in the values of $x_1(t)$ and $x_2(t)$ at every time t. Rather, we are interested in the qualitative properties of $x_1(t)$ and $x_2(t)$. For example, we wish to answer the following question: Do there exist values ξ_1 and ξ_2 at which the two species coexist together in a steady state? That is to say, are there numbers ξ_1, ξ_2 such that $x_1(t) \equiv \xi_1,\ x_2(t) \equiv \xi_2$ a solution of (1)? Such values ξ_1, ξ_2 if they exist, are called **equilibrium points** of (1).

EXAMPLE: Find all equilibrium values of the system of differential equations

$$\frac{dx_1}{dt} = 1 - x_2, \qquad \frac{dx_2}{dt} = x_1^3 + x_2$$

Solution:

$$\mathbf{x^0} = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

is an equilibrium value if, and only if,

$$1 - x_2^0 = 0 \quad \text{and} \quad (x_1^0)^3 + x_2^0 = 0$$

This implies that

$$x_2^0 = 1 \quad \text{and} \quad x_1^0 = -1$$

Hence $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the only equilibrium value of this system.

EXAMPLE: Find all equilibrium values of the system of differential equations

$$\frac{dx}{dt} = (x-1)(y-1), \qquad \frac{dy}{dt} = (x+1)(y+1)$$

Solution:

$$\mathbf{x^0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is an equilibrium value if, and only if,

$$(x_0 - 1)(y_0 - 1) = 0 \quad \text{and} \quad (x_0 + 1)(y_0 + 1) = 0$$

The first equation is satisfied if either x_0 or y_0 is 1, while the second equation is satisfied if either x_0 or y_0 is -1. Hence, $x = 1,\ y = -1$ and $x = -1,\ y = 1$ are the equilibrium solutions of this system.