Complex Roots

If $\lambda = \alpha + i\beta$ is a complex eigenvalue of A with eigenvector $\mathbf{v} = \mathbf{v^1} + i\mathbf{v^2}$, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a complex-valued solution of the differential equation

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

This complex-valued solution gives rise to *two* real-valued solutions, as we now show.

LEMMA 1: Let $\mathbf{x}(t) = \mathbf{y}(t) + i\mathbf{z}(t)$ be a complex-valued solution of (1). Then, both $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are real-valued solutions of (1). Moreover, these two solutions are linearly independent.

EXAMPLE: Solve the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -4x_1 \end{cases} \tag{2}$$

Solution 1: We first note that (2) can be rewritten in the following matrix form or

$$\dot{\mathbf{x}} = \mathbf{Ax}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

This system of equations can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} + 4y = 0 \tag{3}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{dy}{dt}$$

To find two linearly independent solutions of (3) we note that the auxiliary equation is

$$r^2 + 4 = 0$$

with the roots $r_{1,2} = \pm 2i$. Consequently,

$$y_1(t) = e^{0t}\cos 2t = \cos 2t \qquad \text{and} \qquad y_2(t) = e^{0t}\sin 2t = \sin 2t$$

are two solutions of (3). It follows that

$$\mathbf{x^1}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_1'(t) \end{bmatrix} = \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix}$$

and

$$\mathbf{x^2}(t) = \begin{bmatrix} y_2(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

are solutions of (2). The two solutions $\mathbf{x^1}(t)$ and $\mathbf{x^2}(t)$ are linearly independent since their initial values

$$\mathbf{x^1}(0) = \begin{bmatrix} \cos 0 \\ -2\sin 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x^2}(0) = \begin{bmatrix} \sin 0 \\ 2\cos 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

are linearly independent vectors in $\mathbb{R}^2$. Therefore, the general solution $\mathbf{x}(t)$ of (2) is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} y_1(t) \\ y_1'(t) \end{bmatrix} + c_2 \begin{bmatrix} y_2(t) \\ y_2'(t) \end{bmatrix}$$

$$= c_1 \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{bmatrix}$$

Solution 2: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = (-\lambda)^2 - (1)(-4) = \lambda^2 + 4$$

Hence the eigenvalues of A are $\lambda = \pm 2i$. Let $\lambda = 2i$. We use row operations:

$$\begin{bmatrix} -\lambda & 1 & 0 \\ -4 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} -2i & 1 & 0 \\ -4 & -2i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/(2i) & 0 \\ 1 & i/2 & 0 \end{bmatrix}$$

Since

$$-\frac{1}{2i} = -\frac{1 \cdot i}{2i \cdot i} = -\frac{i}{-2} = \frac{i}{2}$$

it follows that

$$\begin{bmatrix} 1 & -1/(2i) & 0 \\ 1 & i/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & i/2 & 0 \\ 1 & i/2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & i/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 + \frac{i}{2}x_2 = 0 \qquad \Longrightarrow \qquad x_1 = -\frac{i}{2}x_2 \qquad \Longrightarrow \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2/2 \\ x_2 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 2i$. Consequently,

$$\mathbf{x}(t) = e^{2it} \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

is a complex-valued solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Now,

$$e^{2it} \begin{bmatrix} -i \\ 2 \end{bmatrix} = e^{2it} \begin{bmatrix} 0 + (-1)i \\ 2 + 0i \end{bmatrix} = (\cos 2t + i \sin 2t) \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$= \cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + i \cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + i \sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2\cos 2t \end{bmatrix} + i \begin{bmatrix} -\cos 2t \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2\sin 2t \end{bmatrix} + \begin{bmatrix} \sin 2t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix} + i \begin{bmatrix} -\cos 2t \\ 2\sin 2t \end{bmatrix}$$

Therefore, by Lemma 1,

$$\mathbf{x^1}(t) = \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix} \qquad \text{and} \qquad \mathbf{x^2}(t) = \begin{bmatrix} -\cos 2t \\ 2\sin 2t \end{bmatrix}$$

are real-valued solutions of (2) and the same result follows.

EXAMPLE: Solve the initial-value problem

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

is

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} &= (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)[(1-\lambda)^2 + 1] \\ &= (1-\lambda)(1 - 2\lambda + \lambda^2 + 1) \\ &= (1-\lambda)(\lambda^2 - 2\lambda + 2) \end{aligned}$$

Hence the eigenvalues of A are

$$\lambda = 1$$

and

$$\begin{aligned} \lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4(-1)}}{2} &= \frac{2 \pm \sqrt{4}\sqrt{-1}}{2} \\ &= \frac{2 \pm 2i}{2} \\ &= 1 \pm i \end{aligned}$$

(a) Let $\lambda = 1$. We use row operations:

$$\left[\begin{array}{ccc|c} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & 1 & 1-\lambda & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \Longrightarrow \qquad \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$$

hence

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. Consequently,

$$\mathbf{x^1}(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is a solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$.

(b) Let $\lambda = 1 + i$. We use row operations:

$$\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & 1 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/i & 0 \\ 0 & 1 & -i & 0 \end{bmatrix}$$

Since $\dfrac{1}{i} = \dfrac{1 \cdot i}{i \cdot i} = \dfrac{i}{-1} = -i$, it follows that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/i & 0 \\ 0 & 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_1 = 0 \\ x_2 - ix_3 = 0 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = ix_3 \end{cases} \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ ix_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1 + i$. Consequently,

$$\mathbf{x}(t) = e^{(1+i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

is a complex-valued solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Now,

$$\begin{aligned} e^{(1+i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} &= e^t e^{it} \begin{bmatrix} 0+0i \\ 0+1i \\ 1+0i \end{bmatrix} = e^t(\cos t + i \sin t) \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= e^t \left(\cos t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \sin t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= e^t \left(\begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix} + i \begin{bmatrix} 0 \\ \cos t \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix} - \begin{bmatrix} 0 \\ \sin t \\ 0 \end{bmatrix} \right) \\ &= e^t \left(\begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix} \right) \\ &= e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + ie^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix} \end{aligned}$$

Therefore, by Lemma 1,

$$\mathbf{x^2}(t) = e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} \qquad \text{and} \qquad \mathbf{x^3}(t) = e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}$$

are real-valued solutions. The three solutions $\mathbf{x^1}(t)$, $\mathbf{x^2}(t)$, and $\mathbf{x^3}(t)$ are linearly independent since their initial values

$$\mathbf{x^1}(0) = e^0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x^2}(0) = e^0 \begin{bmatrix} 0 \\ -\sin 0 \\ \cos 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x^3}(0) = e^0 \begin{bmatrix} 0 \\ \cos 0 \\ \sin 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent vectors in $\mathbb{R}^3$. Therefore, the solution $\mathbf{x}(t)$ of our initial-value problem must have the form

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}$$

Setting $t = 0$, we see that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_3 \\ c_2 \end{bmatrix}$$

Consequently $c_1 = c_2 = c_3 = 1$ and

$$\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix} = e^t \begin{bmatrix} 1 \\ \cos t - \sin t \\ \cos t + \sin t \end{bmatrix}$$