The Eigenvalue-Eigenvector Method of Finding Solutions

EXAMPLE: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear operator such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

Let also

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find $T(\mathbf{x}_1)$, $T(\mathbf{x}_2)$, and $T(\mathbf{x}_3)$.

Solution: We have

$$T(\mathbf{x}_1) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$T(\mathbf{x}_2) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$$

$$T(\mathbf{x}_3) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

EXAMPLE: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear operator such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Find all nonzero vectors $\mathbf{x} \in \mathbb{R}^2$ and all numbers λ such that

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

Solution: Suppose there is a vector $\mathbf{x} \in \mathbb{R}^2$ and a scalar λ such that

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

Since $T(\mathbf{x}) = A\mathbf{x}$, we rewrite this as

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

We have

$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

So, we should find such λ that $(A-\lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Recall that $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det B = 0$ (see the Corollary to Theorem 11 from Section 3.7). From this it follows that

$$A\mathbf{x} = \lambda\mathbf{x}$$

if and only if

$$\det(A - \lambda I) = 0$$

Note that

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}$$

therefore we can rewrite $\det(A - \lambda I) = 0$ as

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$-(3-\lambda)\lambda + 2 = 0 \qquad \Longrightarrow \qquad \lambda^2 - 3\lambda + 2 = 0 \qquad \Longrightarrow \qquad \lambda_1 = 1, \quad \lambda_2 = 2$$

(a) Let $\lambda = 1$. To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - x_2 = 0 \qquad \Longrightarrow \qquad x_1 = x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Let $\lambda = 2$. To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - 2x_2 = 0 \qquad \Longrightarrow \qquad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

<ins>Conclusion</ins>: The equation

$$A\mathbf{x} = \lambda \mathbf{x}$$

has a nonzero solution $\mathbf{x} \in \mathbb{R}^2$ if and only if

$$\lambda = 1 \qquad \text{and} \qquad \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \qquad \text{and} \qquad \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where x_2 is any real number.

DEFINITION: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. A scalar λ is called an **eigenvalue** of A. The set of all solutions of the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

is called the **eigenspace** of A corresponding to λ.

DEFINITION: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

is called the **characteristic polynomial** of A and

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

EXAMPLE: Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Then

$$\begin{vmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{vmatrix} = -(3 - \lambda)\lambda + 2 = \lambda^2 - 3\lambda + 2$$

is a characteristic polynomial,

$$\lambda^2 - 3\lambda + 2 = 0$$

is a characteristic equation; $\lambda = 1$ and $\lambda = 2$ are eigenvalues of A and

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are eigenvectors of A, where t is any real number;

$$\left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of A corresponding to $\lambda = 1$;

$$\left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of A corresponding to $\lambda = 2$.

EXAMPLE: Let

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

Solution: We first solve the following equation:

$$\begin{vmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$(5-\lambda)(1-\lambda) = 0$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A.

(a) Let $\lambda = 1$. To solve the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 = 0$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. The 1-dimensional eigenspace corresponding to $\lambda = 1$ is

$$H = \left\{ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for H.

(b) Let $\lambda = 5$. To solve the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - 2x_2 = 0 \qquad \Longrightarrow \qquad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$. The 1-dimensional eigenspace corresponding to $\lambda = 5$ is

$$H = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for H.

EXAMPLE: Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

Solution: We first solve the following equation:

$$\begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$(\lambda - 1)(\lambda - 5) = 0$$

$$(7-\lambda)(-1-\lambda) - (4)(-3) = -7 - 7\lambda + \lambda + \lambda^2 + 12 = \lambda^2 - 6\lambda + 5$$
$$= (\lambda - 1)(\lambda - 5) = 0$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A.

(a) Let $\lambda = 1$. To solve the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we use row operations:

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 + \frac{2}{3}x_2 = 0 \quad \Longrightarrow \quad x_1 = -\frac{2}{3}x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2/3 \\ x_2 \end{bmatrix} = \frac{1}{3}x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. The 1-dimensional eigenspace corresponding to $\lambda = 1$ is

$$H = \left\{ t \begin{bmatrix} -2 \\ 3 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is a basis for H.

(b) Let $\lambda = 5$. To solve the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we use row operations:

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 + 2x_2 = 0 \quad \Longrightarrow \quad x_1 = -2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$. The 1-dimensional eigenspace corresponding to $\lambda = 5$ is

$$H = \left\{ t \begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for H.

EXAMPLE: Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

The eigenvalues are 2 and 9. Find bases for the corresponding eigenspaces.

Solution:

(a) Let $\lambda = 2$. We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \qquad \Longrightarrow \qquad x_1 = \frac{1}{2}x_2 - 3x_3$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 2$.

To find a basis for the eigenspace corresponding to $\lambda = 2$, we note that

$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot x_2 & + & (-3) \cdot x_3 \\ 1 \cdot x_2 & + & 0 \cdot x_3 \\ 0 \cdot x_2 & + & 1 \cdot x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

therefore the 2-dimensional eigenspace corresponding to $\lambda = 2$ is

$$H = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1,\ t_2 \in \mathbb{R} \right\}$$

and

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for H.

(b) Let $\lambda = 9$. We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 6 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -21 & 21 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \implies \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 9$. The 1-dimensional eigenspace corresponding to $\lambda = 9$ is

$$H = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for H.

THEOREM 12: Any k eigenvectors $\mathbf{v^1}, \dots, \mathbf{v^k}$ of A with distinct eigenvalues $\lambda_1, \dots, \lambda_k$ respectively, are linearly independent.

We return now to the first-order linear homogeneous differential equation

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \tag{1}$$

Our goal is to find n linearly independent solutions $\mathbf{x^1}(t), \ldots, \mathbf{x^n}(t)$. Now, recall that both the first-order and second-order linear homogeneous scalar equations have exponential functions as solutions. This suggests that we try $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a constant vector, as a solution of (1). To this end, observe that

$$\frac{d}{dt}e^{\lambda t}\mathbf{v} = \lambda e^{\lambda t}\mathbf{v}$$

and

$$A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v}$$

Hence, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (1) if, and only if,

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$$

Dividing both sides of this equation by $e^{\lambda t}$ gives

$$A\mathbf{v} = \lambda\mathbf{v}$$

Thus, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (1) if, and only if, λ is an eigenvalue and $\mathbf{v}$ is an eigenvector of A.

EXAMPLE: Find the general solution of the system

$$\begin{cases} x_1' = 7x_1 + 4x_2 \\ x_2' = -3x_1 - x_2 \end{cases} \tag{2}$$

Solution 1: The system can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 5y = 0 \tag{3}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{1}{4}\left(\frac{dy}{dt} - 7y\right)$$

Indeed, if we rewrite the first equation of system (2) as

$$x_2 = \frac{1}{4}\left(x_1' - 7x_1\right)$$

and plug it into the second equation, we get

$$\begin{aligned} \frac{1}{4}\left(x_1' - 7x_1\right)' &= -3x_1 - \frac{1}{4}\left(x_1' - 7x_1\right) \\ \left(x_1' - 7x_1\right)' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 7x_1' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 6x_1' + 5x_1 &= 0 \end{aligned}$$

To find two linearly independent solutions of (3) we note that the auxiliary equation is $r^2 - 6r + 5 = 0$ with the roots $r_1 = 1$ and $r_2 = 5$. Consequently,

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = e^{5t}$$

are two solutions of (3).

We see that

$$\mathbf{x^1}(t) = \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x^2}(t) = \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}$$

We have

$$\mathbf{x^1}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ (y_1'(t) - 7y_1(t))/4 \end{bmatrix} = \begin{bmatrix} e^t \\ (e^t - 7e^t)/4 \end{bmatrix} = \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix}$$

and

$$\mathbf{x^2}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_2(t) \\ (y_2'(t) - 7y_2(t))/4 \end{bmatrix} = \begin{bmatrix} e^{5t} \\ (5e^{5t} - 7e^{5t})/4 \end{bmatrix} = \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}$$

are two solutions of (2). To determine whether $\mathbf{x^1}$ and $\mathbf{x^2}$ are linearly dependent or linearly independent, we check whether their initial values

$$\mathbf{x^1}(0) = \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x^2}(0) = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

are linearly dependent or linearly independent vectors in $\mathbb{R}^2$. Since $\mathbf{x^1}(0)$ and $\mathbf{x^1}(0)$ are not multiples of each other, they are linearly independent. Consequently, by the Theorem (Test for Linear Independence) from Section 3.4, $\mathbf{x^1}(t)$ and $\mathbf{x^2}(t)$ are linearly independent solutions of (2), and every solution $\mathbf{x}(t)$ of (2) can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x^1}(t) + c_2\mathbf{x^2}(t) = c_1 \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix} = \begin{bmatrix} c_1e^t + c_2e^{5t} \\ -3c_1e^t/2 - c_2e^{5t}/2 \end{bmatrix}$$

Solution 2: Note that system (2) can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

The characteristic polynomial of the matrix A is

$$p(\lambda) = \lambda^2 - 6\lambda + 5$$

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} \\ &= (7-\lambda)(-1-\lambda) - (4)(-3) = -7 - 7\lambda + \lambda + \lambda^2 + 12 = \lambda^2 - 6\lambda + 5 \end{aligned}$$

so the eigenvalues of A is $\lambda = 1$ and $\lambda = 5$.

(a) Let $\lambda = 1$. We use row operations:

$$\left[\begin{array}{cc|c} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{array}\right] \sim \underbrace{\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 + \frac{2}{3}x_2 = 0 \qquad \Longrightarrow \qquad x_1 = -\frac{2}{3}x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_2 \\ x_2 \end{bmatrix} = \frac{1}{3}x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. Consequently,

$$ce^t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^1}(t) = e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

(b) Let $\lambda = 5$. We use row operations:

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 + 2x_2 = 0 \qquad \Longrightarrow \qquad x_1 = -2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$. Consequently,

$$ce^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^1}(t) = e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The solutions $\mathbf{x^1}(t)$ and $\mathbf{x^2}(t)$ must be linearly independent, since A has distinct eigenvalues. Therefore, every solution $\mathbf{x}(t)$ must be of the form

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1e^t - 2c_2e^{5t} \\ 3c_1e^t + c_2e^{5t} \end{bmatrix}$$

EXAMPLE: Find all solutions of the equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x}$$

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

is

$$p(\lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

We have

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix} \\ &= (1-\lambda)\Big[(2-\lambda)(-1-\lambda) - (-1)(1)\Big] - (-1)\Big[(3)(-1-\lambda) - (-1)(2)\Big] + 4\Big[(3)(1) - (2-\lambda)(2)\Big] \\ &= (1-\lambda)(-2-2\lambda+\lambda+\lambda^2+1) - (-1)(-3-3\lambda+2) + 4(3-4+2\lambda) \\ &= (1-\lambda)(\lambda^2-\lambda-1) - (-1)(-1-3\lambda) + 4(-1+2\lambda) \\ &= \lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda - 1 - 3\lambda - 4 + 8\lambda \\ &= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \end{aligned}$$

Note that

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = -(\lambda-1)(\lambda-3)(\lambda+2)$$

By inspection, $\lambda = 1$ is a root of $p(\lambda)$, therefore

$$\begin{aligned} -\lambda^3 + 2\lambda^2 + 5\lambda - 6 &= -\lambda^3 + \lambda^2 + \lambda^2 - \lambda + 6\lambda - 6 \\ &= -\lambda^2(\lambda-1) + \lambda(\lambda-1) + 6(\lambda-1) \\ &= (\lambda-1)(-\lambda^2+\lambda+6) \\ &= -(\lambda-1)(\lambda^2-\lambda-6) \\ &= -(\lambda-1)(\lambda-3)(\lambda+2) \end{aligned}$$

so the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 3$, and $\lambda_3 = -2$.

(a) Let $\lambda = 1$. We use row operations:

$$\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - 4x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = 4x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. Consequently,

$$ce^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^1}(t) = e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

(b) Let $\lambda = 3$. We use row operations:

$$\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

$$\left[\begin{array}{cccc} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{array}\right] = \left[\begin{array}{cccc} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \underbrace{\left[\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Echelon Form}}$$

$$\sim \left[\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \underbrace{\left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = x_3 \\ x_2 = 2x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 3$. Consequently,

$$ce^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^2}(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(c) Let $\lambda = -2$. We use row operations:

$$\left[\begin{array}{cccc} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{array}\right] \sim \underbrace{\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Reduced Echelon Form}}$$

$$\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & -5 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = -2$. Consequently,

$$ce^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^3}(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The solutions $\mathbf{x^1}(t),\ \mathbf{x^2}(t),\ \mathbf{x^3}(t)$ must be linearly independent, since A has distinct eigenvalues. Therefore, every solution $x(t)$ must be of the form

$$\mathbf{x}(t) = c_1e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1e^t + c_2e^{3t} - c_3e^{-2t} \\ 4c_1e^t + 2c_2e^{3t} + c_3e^{-2t} \\ c_1e^t + c_2e^{3t} + c_3e^{-2t} \end{bmatrix}$$