

# The Eigenvalue-Eigenvector Method of Finding Solutions

EXAMPLE: Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear operator such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

Let also

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find  $T(\mathbf{x}_1)$ ,  $T(\mathbf{x}_2)$ , and  $T(\mathbf{x}_3)$ .

Solution: We have

$$T(\mathbf{x}_1) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$T(\mathbf{x}_2) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$$

$$T(\mathbf{x}_3) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

EXAMPLE: Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear operator such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Find all nonzero vectors  $\mathbf{x} \in \mathbb{R}^2$  and all numbers  $\lambda$  such that

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

Solution: Suppose there is a vector  $\mathbf{x} \in \mathbb{R}^2$  and a scalar  $\lambda$  such that

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

Since  $T(\mathbf{x}) = A\mathbf{x}$ , we rewrite this as

$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

So, we should find such  $\lambda$  that  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  has a nontrivial solution. Recall that  $B\mathbf{x} = \mathbf{0}$  has a nontrivial solution if and only if  $\det B = 0$  (see the Corollary to Theorem 11 from Section 3.7). From this it follows that

$$A\mathbf{x} = \lambda\mathbf{x}$$

if and only if

$$\det(A - \lambda I) = 0$$

Note that

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{bmatrix}$$

therefore we can rewrite  $\det(A - \lambda I) = 0$  as

$$\begin{vmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$-(3 - \lambda)\lambda + 2 = 0 \quad \implies \quad \lambda^2 - 3\lambda + 2 = 0 \quad \implies \quad \lambda_1 = 1, \quad \lambda_2 = 2$$

(a) Let  $\lambda = 1$ . To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations:

$$\begin{bmatrix} 3 - \lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - x_2 = 0 \quad \implies \quad x_1 = x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Let  $\lambda = 2$ . To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations:

$$\begin{bmatrix} 3 - \lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - 2x_2 = 0 \quad \implies \quad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Conclusion: The equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

has a nonzero solution  $\mathbf{x} \in \mathbb{R}^2$  if and only if

$$\lambda = 1 \quad \text{and} \quad \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \quad \text{and} \quad \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where  $x_2$  is any nonzero real number.

DEFINITION: An **eigenvector** of an  $n \times n$  matrix  $A$  is a nonzero vector  $\mathbf{x}$  such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar  $\lambda$ . A scalar  $\lambda$  is called an **eigenvalue** of  $A$ . The set of all solutions of the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

is called the **eigenspace** of  $A$  corresponding to  $\lambda$ .

DEFINITION: Let  $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ , then

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

is called the **characteristic polynomial** of  $A$  and

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of  $A$ .

EXAMPLE: Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Then

$$\begin{vmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{vmatrix} = -(3 - \lambda)\lambda + 2 = \lambda^2 - 3\lambda + 2$$

is a characteristic polynomial,

$$\lambda^2 - 3\lambda + 2 = 0$$

is a characteristic equation;  $\lambda = 1$  and  $\lambda = 2$  are the eigenvalues of  $A$  and

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are the eigenvectors of  $A$ , where  $t$  is any nonzero real number;

$$\left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of  $A$  corresponding to  $\lambda = 1$ ;

$$\left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of  $A$  corresponding to  $\lambda = 2$ .

EXAMPLE: Let

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

Solution: We first solve the following equation:

$$\begin{vmatrix} 5 - \lambda & 0 \\ 2 & 1 - \lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$(5 - \lambda)(1 - \lambda) = 0$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are the eigenvalues of  $A$ .

(a) Let  $\lambda = 1$ . To solve the homogeneous system  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  we use row operations:

$$\begin{bmatrix} 5 - \lambda & 0 & 0 \\ 2 & 1 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 = 0$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 1$ . The 1-dimensional eigenspace corresponding to  $\lambda = 1$  is

$$H = \left\{ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is a basis for  $H$ .

(b) Let  $\lambda = 5$ . To solve the homogeneous system  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  we use row operations:

$$\begin{bmatrix} 5 - \lambda & 0 & 0 \\ 2 & 1 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \\ \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - 2x_2 = 0 \quad \implies \quad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 5$ . The 1-dimensional eigenspace corresponding to  $\lambda = 5$  is

$$H = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is a basis for  $H$ .

EXAMPLE: Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

Solution: We first solve the following equation:

$$\begin{vmatrix} 7 - \lambda & 4 \\ -3 & -1 - \lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$\begin{aligned} (7 - \lambda)(-1 - \lambda) - (4)(-3) &= -7 - 7\lambda + \lambda + \lambda^2 + 12 = \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 1)(\lambda - 5) = 0 \end{aligned}$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are the eigenvalues of  $A$ .

(a) Let  $\lambda = 1$ . To solve the homogeneous system  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  we use row operations:

$$\begin{aligned} \begin{bmatrix} 7 - \lambda & 4 & 0 \\ -3 & -1 - \lambda & 0 \end{bmatrix} &= \begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \\ &\sim \underbrace{\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}} \end{aligned}$$

hence

$$x_1 + \frac{2}{3}x_2 = 0 \quad \implies \quad x_1 = -\frac{2}{3}x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2/3 \\ x_2 \end{bmatrix} = \frac{1}{3}x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 1$ . The 1-dimensional eigenspace corresponding to  $\lambda = 1$  is

$$H = \left\{ t \begin{bmatrix} -2 \\ 3 \end{bmatrix} : t \text{ is any real number} \right\}$$

and  $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$  is a basis for  $H$ .

(b) Let  $\lambda = 5$ . To solve the homogeneous system  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  we use row operations:

$$\begin{bmatrix} 7 - \lambda & 4 & 0 \\ -3 & -1 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 + 2x_2 = 0 \implies x_1 = -2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 5$ . The 1-dimensional eigenspace corresponding to  $\lambda = 5$  is

$$H = \left\{ t \begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  is a basis for  $H$ .

EXAMPLE: Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

The eigenvalues are 2 and 9. Find bases for the corresponding eigenspaces.

Solution:

(a) Let  $\lambda = 2$ . We use row operations:

$$\begin{bmatrix} 4 - \lambda & -1 & 6 & 0 \\ 2 & 1 - \lambda & 6 & 0 \\ 2 & -1 & 8 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \implies x_1 = \frac{1}{2}x_2 - 3x_3$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 2$ .

To find a basis for the eigenspace corresponding to  $\lambda = 2$ , we note that

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot x_2 + (-3) \cdot x_3 \\ 1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_2 + 1 \cdot x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

therefore the 2-dimensional eigenspace corresponding to  $\lambda = 2$  is

$$H = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

and

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for  $H$ .

(b) Let  $\lambda = 9$ . We use row operations:

$$\begin{aligned} \begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} &= \begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 6 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -21 & 21 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}} \end{aligned}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \implies \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 9$ . The 1-dimensional eigenspace corresponding to  $\lambda = 9$  is

$$H = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is a basis for  $H$ .

**THEOREM 12:** Any  $k$  eigenvectors  $\mathbf{v}^1, \dots, \mathbf{v}^k$  of  $A$  with distinct eigenvalues  $\lambda_1, \dots, \lambda_k$  respectively, are linearly independent.

We return now to the first-order linear homogeneous differential equation

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \quad (1)$$

Our goal is to find  $n$  linearly independent solutions  $\mathbf{x}^1(t), \dots, \mathbf{x}^n(t)$ . Now, recall that both the first-order and second-order linear homogeneous scalar equations have exponential functions as solutions. This suggests that we try  $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$ , where  $\mathbf{v}$  is a constant vector, as a solution of (1). To this end, observe that

$$\frac{d}{dt} e^{\lambda t} \mathbf{v} = \lambda e^{\lambda t} \mathbf{v}$$

and

$$A(e^{\lambda t} \mathbf{v}) = e^{\lambda t} A \mathbf{v}$$

Hence,  $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$  is a solution of (1) if, and only if,

$$\lambda e^{\lambda t} \mathbf{v} = e^{\lambda t} A \mathbf{v}$$

Dividing both sides of this equation by  $e^{\lambda t}$  gives

$$A \mathbf{v} = \lambda \mathbf{v}$$

Thus,  $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$  is a solution of (1) if, and only if,  $\lambda$  is an eigenvalue and  $\mathbf{v}$  is an eigenvector of  $A$ .

EXAMPLE: Find the general solution of the system

$$\begin{cases} x_1' = 7x_1 + 4x_2 \\ x_2' = -3x_1 - x_2 \end{cases} \quad (2)$$

Solution 1: The system can be derived from the second-order differential equation

$$\frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 5y = 0 \quad (3)$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{1}{4} \left( \frac{dy}{dt} - 7y \right)$$

Indeed, if we rewrite the first equation of system (2) as

$$x_2 = \frac{1}{4} (x_1' - 7x_1)$$

and plug it into the second equation, we get

$$\begin{aligned} \frac{1}{4} (x_1' - 7x_1)' &= -3x_1 - \frac{1}{4} (x_1' - 7x_1) \\ (x_1' - 7x_1)' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 7x_1' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 6x_1' + 5x_1 &= 0 \end{aligned}$$

To find two linearly independent solutions of (3) we note that the characteristic equation is  $r^2 - 6r + 5 = 0$  with the roots  $r_1 = 1$  and  $r_2 = 5$ . Consequently,

$$y_1(t) = e^t \quad \text{and} \quad y_2(t) = e^{5t}$$

are two solutions of (3).



We see that

$$\mathbf{x}^1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ (y_1'(t) - 7y_1(t))/4 \end{bmatrix} = \begin{bmatrix} e^t \\ (e^t - 7e^t)/4 \end{bmatrix} = \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix}$$

and

$$\mathbf{x}^2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_2(t) \\ (y_2'(t) - 7y_2(t))/4 \end{bmatrix} = \begin{bmatrix} e^{5t} \\ (5e^{5t} - 7e^{5t})/4 \end{bmatrix} = \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}$$

are two solutions of (2). To determine whether  $\mathbf{x}^1$  and  $\mathbf{x}^2$  are linearly dependent or linearly independent, we check whether their initial values

$$\mathbf{x}^1(0) = \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^2(0) = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

are linearly dependent or linearly independent vectors in  $\mathbb{R}^2$ . Since  $\mathbf{x}^1(0)$  and  $\mathbf{x}^2(0)$  are not multiples of each other, they are linearly independent. Consequently, by the Theorem (Test for Linear Independence) from Section 3.4,  $\mathbf{x}^1(t)$  and  $\mathbf{x}^2(t)$  are linearly independent solutions of (2), and every solution  $\mathbf{x}(t)$  of (2) can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) = c_1 \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix} = \begin{bmatrix} c_1e^t + c_2e^{5t} \\ -3c_1e^t/2 - c_2e^{5t}/2 \end{bmatrix}$$

Solution 2: Note that system (2) can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

The characteristic polynomial of the matrix  $A$  is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 7 - \lambda & 4 \\ -3 & -1 - \lambda \end{vmatrix} \\ &= (7 - \lambda)(-1 - \lambda) - (4)(-3) = -7 - 7\lambda + \lambda + \lambda^2 + 12 = \lambda^2 - 6\lambda + 5 \end{aligned}$$

so the eigenvalues of  $A$  are  $\lambda = 1$  and  $\lambda = 5$ .

(a) Let  $\lambda = 1$ . We use row operations:

$$\begin{aligned} \begin{bmatrix} 7 - \lambda & 4 & 0 \\ -3 & -1 - \lambda & 0 \end{bmatrix} &= \begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \\ &\sim \underbrace{\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}} \end{aligned}$$

hence

$$x_1 + \frac{2}{3}x_2 = 0 \quad \implies \quad x_1 = -\frac{2}{3}x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_2 \\ x_2 \end{bmatrix} = \frac{1}{3}x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 1$ . Consequently,

$$ce^t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is a solution of the differential equation for any constant  $c$ . For simplicity, we take

$$\mathbf{x}^1(t) = e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

(b) Let  $\lambda = 5$ . We use row operations:

$$\begin{bmatrix} 7-\lambda & 4 & 0 \\ -3 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 + 2x_2 = 0 \quad \implies \quad x_1 = -2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 5$ . Consequently,

$$ce^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant  $c$ . For simplicity, we take

$$\mathbf{x}^2(t) = e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The solutions  $\mathbf{x}^1(t)$  and  $\mathbf{x}^2(t)$  must be linearly independent, since  $A$  has distinct eigenvalues. Therefore, every solution  $\mathbf{x}(t)$  must be of the form

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 e^t - 2c_2 e^{5t} \\ 3c_1 e^t + c_2 e^{5t} \end{bmatrix}$$

EXAMPLE: Find all solutions of the equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x}$$

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

is

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \begin{vmatrix} 1 - \lambda & -1 & 4 \\ 3 & 2 - \lambda & -1 \\ 2 & 1 & -1 - \lambda \end{vmatrix} \\
&= (1 - \lambda) \begin{vmatrix} 2 - \lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} - (-1) \begin{vmatrix} 3 & -1 \\ 2 & -1 - \lambda \end{vmatrix} + 4 \begin{vmatrix} 3 & 2 - \lambda \\ 2 & 1 \end{vmatrix} \\
&= (1 - \lambda) \left[ (2 - \lambda)(-1 - \lambda) - (-1)(1) \right] - (-1) \left[ (3)(-1 - \lambda) - (-1)(2) \right] + 4 \left[ (3)(1) - (2 - \lambda)(2) \right] \\
&= (1 - \lambda)(-2 - 2\lambda + \lambda + \lambda^2 + 1) - (-1)(-3 - 3\lambda + 2) + 4(3 - 4 + 2\lambda) \\
&= (1 - \lambda)(\lambda^2 - \lambda - 1) - (-1)(-1 - 3\lambda) + 4(-1 + 2\lambda) \\
&= \lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda - 1 - 3\lambda - 4 + 8\lambda \\
&= -\lambda^3 + 2\lambda^2 + 5\lambda - 6
\end{aligned}$$

By inspection,  $\lambda = 1$  is a root of  $p(\lambda)$ , therefore

$$\begin{aligned}
-\lambda^3 + 2\lambda^2 + 5\lambda - 6 &= -\lambda^3 + \lambda^2 + \lambda^2 - \lambda + 6\lambda - 6 \\
&= -\lambda^2(\lambda - 1) + \lambda(\lambda - 1) + 6(\lambda - 1) \\
&= (\lambda - 1)(-\lambda^2 + \lambda + 6) \\
&= -(\lambda - 1)(\lambda^2 - \lambda - 6) \\
&= -(\lambda - 1)(\lambda - 3)(\lambda + 2)
\end{aligned}$$

so the eigenvalues of  $A$  are  $\lambda_1 = 1$ ,  $\lambda_2 = 3$ , and  $\lambda_3 = -2$ .

(a) Let  $\lambda = 1$ . We use row operations:

$$\begin{aligned}
\begin{bmatrix} 1 - \lambda & -1 & 4 & 0 \\ 3 & 2 - \lambda & -1 & 0 \\ 2 & 1 & -1 - \lambda & 0 \end{bmatrix} &= \begin{bmatrix} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \\
&\sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}
\end{aligned}$$

hence

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - 4x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = 4x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 1$ . Consequently,

$$ce^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant  $c$ . For simplicity, we take

$$\mathbf{x}^1(t) = e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

(b) Let  $\lambda = 3$ . We use row operations:

$$\begin{aligned} \begin{bmatrix} 1 - \lambda & -1 & 4 & 0 \\ 3 & 2 - \lambda & -1 & 0 \\ 2 & 1 & -1 - \lambda & 0 \end{bmatrix} &= \begin{bmatrix} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\sim \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}} \end{aligned}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \implies \begin{cases} x_1 = x_3 \\ x_2 = 2x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = 3$ . Consequently,

$$ce^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant  $c$ . For simplicity, we take

$$\mathbf{x}^2(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(c) Let  $\lambda = -2$ . We use row operations:

$$\begin{aligned}
 \begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} &= \begin{bmatrix} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & -5 & 0 \end{bmatrix} \\
 &\sim \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \\
 &\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 &\sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}
 \end{aligned}$$

hence

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of  $A$ , corresponding to  $\lambda = -2$ . Consequently,

$$ce^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant  $c$ . For simplicity, we take

$$\mathbf{x}^3(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The solutions  $\mathbf{x}^1(t)$ ,  $\mathbf{x}^2(t)$ ,  $\mathbf{x}^3(t)$  must be linearly independent, since  $A$  has distinct eigenvalues. Therefore, every solution  $x(t)$  must be of the form

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 e^t + c_2 e^{3t} - c_3 e^{-2t} \\ 4c_1 e^t + 2c_2 e^{3t} + c_3 e^{-2t} \\ c_1 e^t + c_2 e^{3t} + c_3 e^{-2t} \end{bmatrix}$$