The Eigenvalue-Eigenvector Method of Finding Solutions

EXAMPLE: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear operator such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

Let also

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find $T(\mathbf{x}_1)$, $T(\mathbf{x}_2)$, and $T(\mathbf{x}_3)$.

Solution: We have

$$T(\mathbf{x}_1) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$T(\mathbf{x}_2) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$$

$$T(\mathbf{x}_3) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

EXAMPLE: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear operator such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Find a nonzero vector $\mathbf{x} \in \mathbb{R}^2$ such that

$$T(\mathbf{x}) = \lambda \mathbf{x}$$

Solution: Suppose there is a vector $\mathbf{x} \in \mathbb{R}^2$ and a scalar λ such that

$$T(\mathbf{x}) = \lambda \mathbf{x}$$

Since $T(\mathbf{x}) = A\mathbf{x}$, we rewrite this as

$$A\mathbf{x} = \lambda \mathbf{x}$$

$$A\mathbf{x} - \lambda \mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - \lambda I \mathbf{x} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

So, we should find such λ that $(A-\lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Recall that $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det B = 0$. From this it follows that

$$A\mathbf{x} = \lambda \mathbf{x}$$

if and only if

$$\det(A - \lambda I) = 0$$

Note that

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}$$

therefore we can rewrite $\det(A - \lambda I) = 0$ as

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$-(3-\lambda)\lambda + 2 = 0 \qquad \Longrightarrow \qquad \lambda^2 - 3\lambda + 2 = 0 \qquad \Longrightarrow \qquad \lambda_1 = 1, \quad \lambda_2 = 2$$

(a) Let $\lambda = 1$. To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - x_2 = 0 \qquad \Longrightarrow \qquad x_1 = x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Let $\lambda = 2$. To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - 2x_2 = 0 \qquad \Longrightarrow \qquad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Conclusion: The equation

$$A\mathbf{x} = \lambda \mathbf{x}$$

has a nonzero solution $\mathbf{x} \in \mathbb{R}^2$ if and only if

$$\lambda = 1 \qquad \text{and} \qquad \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \qquad \text{and} \qquad \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where x_2 is any real number.

DEFINITION: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ. A scalar λ is called an **eigenvalue** of A.

EXAMPLE: Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Then $\lambda = 1$ and $\lambda = 2$ are eigenvalues of A and

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are eigenvectors of A, where t is any real number.

DEFINITION: Let A be an $n \times n$ matrix and let λ be an eigenvalue. The set of all solutions of the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

is called the **eigenspace** of A corresponding to λ.

EXAMPLE: Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Then

$$\left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of A corresponding to $\lambda = 1$;

$$\left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of A corresponding to $\lambda = 2$.

EXAMPLE: Let

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

Solution: We first solve the following equation:

$$\begin{vmatrix} 5 - \lambda & 0 \\ 2 & 1 - \lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$(5 - \lambda)(1 - \lambda) = 0$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A.

(a) Let $\lambda = 1$. To solve the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 = 0$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. The 1-dimensional eigenspace corresponding to $\lambda = 1$ is

$$H = \left\{ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the basis for H.

(b) Let $\lambda = 5$. To solve the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - 2x_2 = 0 \qquad \Longrightarrow \qquad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$. The 1-dimensional eigenspace corresponding to $\lambda = 5$ is

$$H = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the basis for H.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

is called the **characteristic polynomial** of A and

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

EXAMPLE: Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

The eigenvalues are 2 and 9. Find bases for the corresponding eigenspaces.

Solution:

(a) Let $\lambda = 2$. We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \qquad \Longrightarrow \qquad x_1 = \frac{1}{2}x_2 - 3x_3$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 2$.

To find a basis for the eigenspace corresponding to $\lambda = 2$, we note that

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 + (-3)x_3 \\ 1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_2 + 1 \cdot x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

therefore the 2-dimensional eigenspace corresponding to $\lambda = 2$ is

$$H = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1,\ t_2 \in \mathbb{R} \right\}$$

and

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the basis for H.

(b) Let $\lambda = 9$. We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 6 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -21 & 21 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 9$. The 1-dimensional eigenspace corresponding to $\lambda = 9$ is

$$H = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the basis for H.

THEOREM 12: Any k eigenvectors $\mathbf{v^1}, \ldots, \mathbf{v^k}$ of A with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, are linearly independent.

We return now to the first-order linear homogeneous differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \tag{1}$$

Our goal is to find n linearly independent solutions $\mathbf{x^1}(t), \ldots, \mathbf{x^n}(t)$. Now, recall that both the first-order and second-order linear homogeneous scalar equations have exponential functions as solutions. This suggests that we try $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a constant vector, as a solution of (1). To this end, observe that

$$\frac{d}{dt} e^{\lambda t}\mathbf{v} = \lambda e^{\lambda t}\mathbf{v}$$

and

$$A(e^{\lambda t}\mathbf{v}) = e^{\lambda t} A\mathbf{v}$$

Hence, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (1) if, and only if,

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t} A\mathbf{v}$$

Dividing both sides of this equation by $e^{\lambda t}$ gives

$$A\mathbf{v} = \lambda \mathbf{v}$$

Thus, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (1) if, and only if, λ is an eigenvalue and $\mathbf{v}$ is an eigenvector of A.

EXAMPLE: Find all solutions of the equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x}$$

Solution: The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

is

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix} \\
&= (1-\lambda)\Big[(2-\lambda)(-1-\lambda) - (-1)(1)\Big] - (-1)\Big[(3)(-1-\lambda) - (-1)(2)\Big] + 4\Big[(3)(1) - (2-\lambda)(2)\Big] \\
&= (1-\lambda)(-2 - 2\lambda + \lambda + \lambda^2 + 1) - (-1)(-3 - 3\lambda + 2) + 4(3 - 4 + 2\lambda) \\
&= (1-\lambda)(\lambda^2 - \lambda - 1) - (-1)(-1 - 3\lambda) + 4(-1 + 2\lambda) \\
&= \lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda - 1 - 3\lambda - 4 + 8\lambda \\
&= -\lambda^3 + 2\lambda^2 + 5\lambda - 6
\end{aligned}$$

By inspection, $\lambda = 1$ is a root of $p(\lambda)$, therefore

$$\begin{aligned}
-\lambda^3 + 2\lambda^2 + 5\lambda - 6 &= -\lambda^3 + \lambda^2 + \lambda^2 - \lambda + 6\lambda - 6 \\
&= -\lambda^2(\lambda - 1) + \lambda(\lambda - 1) + 6(\lambda - 1) \\
&= (\lambda - 1)(-\lambda^2 + \lambda + 6) \\
&= -(\lambda - 1)(\lambda^2 - \lambda - 6) \\
&= -(\lambda - 1)(\lambda - 3)(\lambda + 2)
\end{aligned}$$

so the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 3$, and $\lambda_3 = -2$.

(a) Let $\lambda = 1$. We use row operations:

$$\begin{aligned}
\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} &\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \\
&\sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}
\end{aligned}$$

hence

$$\begin{cases} x_1 + x_3 = 0 \\ \\ x_2 - 4x_3 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = -x_3 \\ \\ x_2 = 4x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. Consequently,

$$ce^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^1}(t) = e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

(b) Let $\lambda = 3$. We use row operations:

$$\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ \\ x_2 - 2x_3 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = x_3 \\ \\ x_2 = 2x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 3$. Consequently,

$$ce^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^2}(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(c) Let $\lambda = -2$. We use row operations:

$$\begin{bmatrix} 1-\lambda & -1 & 4 & 0 \\ 3 & 2-\lambda & -1 & 0 \\ 2 & 1 & -1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & -5 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

hence

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \end{cases}$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = -2$. Consequently,

$$ce^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is a solution of the differential equation for any constant c. For simplicity, we take

$$\mathbf{x^3}(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The solutions $\mathbf{x^1}(t),\ \mathbf{x^2}(t),\ \mathbf{x^3}(t)$ must be linearly independent, since A has distinct eigenvalues. Therefore, every solution $x(t)$ must be of the form

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 e^t + c_2 e^{3t} - c_3 e^{-2t} \\ 4c_1 e^t + 2c_2 e^{3t} + c_3 e^{-2t} \\ c_1 e^t + c_2 e^{3t} + c_3 e^{-2t} \end{bmatrix}$$