Linear Transformations

Recall that a system of linear equations can be rewritten in an equivalent matrix form. For example, the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 17x_4 = 18 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 = 57 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 = 19 \end{cases}$$

can be rewritten as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 18 \\ 57 \\ 19 \end{bmatrix}$$

Indeed,

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}$$

and $A\mathbf{x}$ is equal to $\mathbf{b}$ if and only if the corresponding components are equal.

In the previous section we approached the problem of solving the equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

by asking whether the matrix A^{-1} exists. This approach led us to the conclusion that equation (1) has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

if

$$\det A \neq 0$$

In order to determine what happens when $\det A = 0$, we will approach the problem of solving (1) in an entirely different manner. We begin with the following lemma.

LEMMA 1: Let A be an $n \times n$ matrix with elements a_{ij}, and let $\mathbf{x}$ be a vector with components $x_1,\ x_2, \dots, x_n$. Let

$$\mathbf{a^j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

denote the jth column of A. Then

$$A\mathbf{x} = x_1\mathbf{a^1} + x_2\mathbf{a^2} + \dots + x_n\mathbf{a^n}$$

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then

$$\begin{aligned} A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 17 \\ 46 \\ 31 \end{bmatrix} \\ &= x_1\mathbf{a^1} + x_2\mathbf{a^2} + x_3\mathbf{a^3} + x_4\mathbf{a^4} \end{aligned}$$

where

$$\mathbf{a^1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{a^2} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \quad \mathbf{a^3} = \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix}, \quad \mathbf{a^4} = \begin{bmatrix} 17 \\ 46 \\ 31 \end{bmatrix}$$

THEOREM 9:

(a) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution if the columns of A are linearly independent.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has either no solution, or infinitely many solutions, if the columns of A are linearly dependent.

EXAMPLE:

(a) For which vectors

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

can we solve the equation

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{x} = \mathbf{b}?$$

Solution: The columns of the matrix A are

$$\mathbf{a^1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a^2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a^3} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

By Lemma 1,

$$A\mathbf{x} = c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^3}$$

for some constants c_1', c_2', and c_3'. Since $A\mathbf{x} = \mathbf{b}$, it follows that this system has a solution if, and only if,

$$\mathbf{b} = c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^3}$$

Now note that $\mathbf{a^1}$ and $\mathbf{a^2}$ are linearly independent while $\mathbf{a^3}$ is the sum of $\mathbf{a^1}$ and $\mathbf{a^2}$. Hence, we can solve the equation $A\mathbf{x} = \mathbf{b}$ if, and only if,

$$\begin{aligned}
\mathbf{b} &= c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^3} \\
&= c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'(\mathbf{a^1} + \mathbf{a^2}) \\
&= c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^1} + c_3'\mathbf{a^2} \\
&= \underbrace{(c_1' + c_3')}_{c_1}\mathbf{a^1} + \underbrace{(c_2' + c_3')}_{c_2}\mathbf{a^2} \\
&= c_1\mathbf{a^1} + c_2\mathbf{a^2} \\
&= c_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \\ 2c_1 + c_2 \end{bmatrix}
\end{aligned}$$

for some constants c_1 and c_2. Equivalently, one can show that $b_3 = b_1 + b_2$.

(b) Find all solutions of the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: Consider the corresponding system of three linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 &= 0 \\ x_1 \qquad + x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{cases}$$

Notice that the third equation is the sum of the first two equations. Hence, we need only consider the first two equations. The second equation says that $x_1 = -x_3$. Substituting this value of x_1, into the first equation gives $x_2 = -x_3$. Hence, all solutions of the equation $A\mathbf{x} = \mathbf{0}$ are of the form

$$\mathbf{x} = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(c) Find all solutions of the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution 1: Observe first that

$$\mathbf{x^1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is clearly one solution of this equation. Next, let $\mathbf{x^2}$ be any other solution of this equation. We show that

$$\mathbf{x^2} = \mathbf{x^1} + \xi$$

where ξ is a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Indeed, since $\mathbf{x^1}$ and $\mathbf{x^2}$ are solutions of $A\mathbf{x} = \mathbf{b}$, we get $A\mathbf{x^1} = \mathbf{b}$ and $A\mathbf{x^2} = \mathbf{b}$, therefore

$$\mathbf{0} = \mathbf{b} - \mathbf{b} = A\mathbf{x^2} - A\mathbf{x^1} = A(\mathbf{x^2} - \mathbf{x^1})$$

This means that $\mathbf{x^2} - \mathbf{x^1}$ is a solution of $A\mathbf{x} = \mathbf{0}$. Putting $\xi = \mathbf{x^2} - \mathbf{x^1}$, we get the desired result. Moreover, the sum of any solution of the nonhomogeneous equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

with a solution of the homogeneous equation is again a solution of the nonhomogeneous equation. Indeed, if $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$ and ξ is a solution of $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} + \xi$ is a solution of $A\mathbf{x} = \mathbf{b}$ again, since

$$A(\mathbf{x} + \xi) = A\mathbf{x} + A\xi = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Hence, any solution of the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Solution 2: We have

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 \qquad + x_3 = 0 \\ \qquad x_2 + x_3 = 1 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = 1 - x_3 \\ x_3 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 + (-1) \cdot x_3 \\ 1 + (-1) \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} (-1) \cdot x_3 \\ (-1) \cdot x_3 \\ 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS:

1. Add a multiple of one row to another.

2. Multiply a row by a nonzero constant.

3. Interchange two rows.

ECHELON FORM:

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

REDUCED ECHELON FORM:

$$\begin{bmatrix} 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

DEFINITION: A **pivot position** in a matrix is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position. The variables corresponding to pivot columns are called **basic variables**. The other variables are called **free variables**.

EXAMPLE: Find all solutions of the system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

Solution: We use row operations

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + 3x_2 = -5 \\ x_3 = 3 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \\ x_2 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 3x_2 \\ x_2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 + (-3) \cdot x_2 \\ 0 + 1 \cdot x_2 \\ 3 + 0 \cdot x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Solution: We use row operations

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases}$$

that is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3} \cdot x_3 \\ 2 + 0 \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution: We use row operations

$$\left[\begin{array}{rrrrrr} -3 & 6 & -1 & 1 & -7 & 0\\ 1 & -2 & 2 & 3 & -1 & 0\\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0\\ -3 & 6 & -1 & 1 & -7 & 0\\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0\\ 0 & 0 & 5 & 10 & -10 & 0\\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0\\ 0 & 0 & 1 & 2 & -2 & 0\\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\sim \underbrace{\left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0\\ 0 & 0 & 1 & 2 & -2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Echelon Form}}$$

$$\sim \underbrace{\left[\begin{array}{rrrrrr} 1 & -2 & 0 & -1 & 3 & 0\\ 0 & 0 & 1 & 2 & -2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5\\ x_2\\ -2x_4 + 2x_5\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 2\cdot x_2 + 1\cdot x_4 + (-3)\cdot x_5\\ 1\cdot x_2 + 0\cdot x_4 + 0\cdot x_5\\ 0\cdot x_2 + (-2)\cdot x_4 + 2\cdot x_5\\ 0\cdot x_2 + 1\cdot x_4 + 0\cdot x_5\\ 0\cdot x_2 + 0\cdot x_4 + 1\cdot x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2\\ 1\\ 0\\ 0\\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1\\ 0\\ -2\\ 1\\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3\\ 0\\ 2\\ 0\\ 1 \end{bmatrix}$$

Theorem 9 is an extremely useful theorem since it establishes necessary and sufficient conditions for the equation $A\mathbf{x} = \mathbf{b}$ to have a unique solution. However, it is often very difficult to apply Theorem 9 since it is usually quite difficult to determine whether n vectors are linearly dependent or linearly independent. Fortunately, we can relate the question of whether the columns of A are linearly dependent or linearly independent to the much simpler problem of determining whether the determinant of A is zero or nonzero. There are several different ways of accomplishing this. In this section, we will present a very elegant method which utilizes the important concept of a linear transformation.

DEFINITION: A **transformation** (or **function**, or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ from $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain**, and $\mathbb{R}^m$ is called the **codomain** of T.

EXAMPLE: Let

$$A = \left[\begin{array}{rr} 1 & -3\\ 3 & 5\\ -1 & 7 \end{array}\right], \quad \mathbf{u} = \left[\begin{array}{r} 2\\ -1 \end{array}\right]$$

and let

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T(\mathbf{x}) = A\mathbf{x}$$

Find $T(\mathbf{u})$.

Solution: We have:

$$T(\mathbf{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

DEFINITION: A transformation T is **linear** if:

$$(i) \quad T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \text{ in the domain of } T$$

$$(ii) \quad T(c\mathbf{u}) = cT(\mathbf{u}) \quad \text{for all } \mathbf{u} \text{ in the domain of } T \text{ and all scalars } c$$

THEOREM: If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

EXAMPLE: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation such that

$$T(\mathbf{x}) = A\mathbf{x}$$

where A is an $m \times n$ matrix. Then T is linear, since

$$T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$$

THEOREM 10: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$:

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)]$$

where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$, that is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE: Find a matrix A such that for any $\mathbf{x}$ from $\mathbb{R}^2$ we have

$$T(\mathbf{x}) = 3\mathbf{x}$$

Solution 1: We have

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e}_2) = 3\mathbf{e}_2 = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

therefore

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution 2: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$T(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \qquad \text{and} \qquad 3\mathbf{x} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

Hence,

$$a = 3, \ b = 0, \ c = 0, \ d = 3$$

so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow T(\mathbf{x}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is linear, since

$$T(\mathbf{x}) = I_n\mathbf{x}$$

where I_n is the $n \times n$ identity matrix.

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \\ x_1 \end{bmatrix}$$

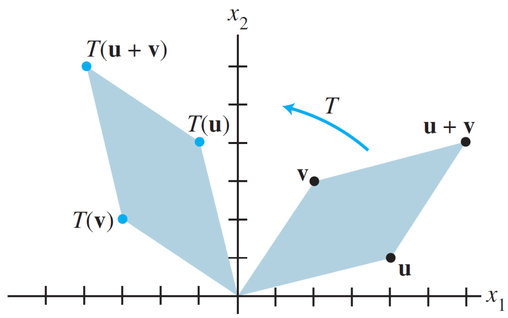
is linear, since

$$T(\mathbf{x}) = A\mathbf{x}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

EXAMPLE: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ϕ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear.



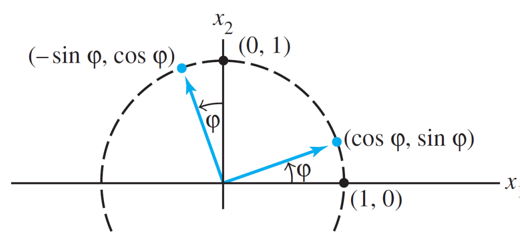
Find the standard matrix A of this transformation.

Solution: Note that

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{rotates into} \quad \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

and

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{rotates into} \quad \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$



By Theorem 10,

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \longrightarrow \quad T(\mathbf{x}) = \begin{bmatrix} 1 \\ x_1^2 + x_2^2 \end{bmatrix}$$

is *not* linear. Indeed, since

$$T(2\mathbf{x}) = \begin{bmatrix} 1 \\ (2x_1)^2 + (2x_2)^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4x_1^2 + 4x_2^2 \end{bmatrix}$$

and

$$2T(\mathbf{x}) = \begin{bmatrix} 2 \\ 2x_1^2 + 2x_2^2 \end{bmatrix}$$

Since

$$T(2\mathbf{x}) \neq 2T(\mathbf{x})$$

it follows that T is *not* linear.

DEFINITION: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

We call S the **inverse** of T and write it as T^{-1}.

LEMMA 2: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by

$$S(\mathbf{x}) = A^{-1}\mathbf{x}$$

is the unique function satisfying both equations

$$S(T(\mathbf{x})) = \mathbf{x} \qquad \text{and} \qquad T(S(\mathbf{x})) = \mathbf{x}$$

We are now ready to relate the problem of determining whether the columns of an $n \times n$ matrix A are linearly dependent or linearly independent to the much simpler problem of determining whether the determinant of A is zero or nonzero.

LEMMA 3: The columns of an $n \times n$ matrix A are linearly independent if, and only if, $\det A \neq 0$.

Proof: We prove Lemma 3 by the following complex, but very clever argument.

(1) The columns of A are linearly independent if, and only if, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every $\mathbf{b}$ in $\mathbb{R}^n$. This statement is just a reformulation of Theorem 9.

(2) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every $\mathbf{b}$ if, and only if, the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ has an inverse.

(3) From Lemma 2, the linear transformation T has an inverse if, and only if, the matrix A^{-1} exists.

(4) Finally, the matrix A^{-1} exists if, and only if, $\det A \neq 0$. This is the content of Theorem 8, Section 3.6. Therefore, we conclude that the columns of A are linearly independent if, and only if, $\det A \neq 0$. ■

EXAMPLE: Show that vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent. Then show that vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent.

Solution: We have

$$\det[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Since the determinant is equal to zero, the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent by Lemma 3 above. Similarly,

$$\det[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Since the determinant is not equal to zero, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent by Lemma 3 above.

REMARK: One can show that the following stronger statement is true: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ from $\mathbb{R}^n$ are linearly independent if, and only if, an echelon form of the matrix

$$A = [\mathbf{v}_1 \ldots \mathbf{v}_m]$$

has a pivot in every column; these vectors are dependent otherwise. Moreover, the pivot columns of the matrix A form a basis for the vector space spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

EXAMPLE: Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the same as above. We have

$$A = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (two) is lesser than the number of columns (three), the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent. Moreover, the vectors $\mathbf{u}_1, \mathbf{u}_2$ (the pivot columns of the matrix A) form a basis for the vector space spanned by the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. So,

$$\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$$

Furthermore, by Theorem 2 from Section 3.3, the dimension of $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ (and therefore the dimension of $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$) is *two*, since $\mathbf{u}_1, \mathbf{u}_2$ are *two* linearly independent vectors.

Similarly,

$$B = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (three) is equal to the number of columns (three), the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Moreover, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (the pivot columns of the matrix B) form a basis for the vector space spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Furthermore, by Theorem 3 from Section 3.3, these vectors form a basis for $\mathbb{R}^3$ (and therefore $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$), since these *three* vectors are linearly independent and the dimension of $\mathbb{R}^3$ is also *three*.

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix}$$

(a) Find a basis and the dimension of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

(b) Find all solutions of the system $A\mathbf{x} = \mathbf{0}$, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5]$.

(c) Find a basis and the dimension of the vector space of the solutions of the system $A\mathbf{x} = \mathbf{0}$.

Solution:

(a) We have

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since column 1 and column 2 are the pivot columns, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

form a basis for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. The dimension of this vector space is 2.

(b) We have

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 & 0 \\ 2 & 1 & 1 & -1 & 0 & 0 \\ 0 & 9 & -3 & -1 & -6 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 2 & 0 & 3 & 0 \\ 0 & 9 & -3 & -1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ x_2 = \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \end{cases}$$

therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{u}_3}$$

(c) One can check that the vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent. Indeed,

$$[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{bmatrix} -\frac{2}{3} & \frac{4}{9} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{9} & \frac{2}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{2}{3} & \frac{4}{9} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{9} & \frac{2}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since each column has a pivot, vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent. Therefore they form a basis for the vector space of the solutions of the system $A\mathbf{x} = \mathbf{0}$. The dimension of this vector space is 3.

REMARK: In other words, the dimension of Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is the number of *pivot* columns of an echelon form of $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5]$. To the contrast, the dimension of the vector space of the solutions of the system $A\mathbf{x} = \mathbf{0}$ is the number of *non-pivot* columns of an echelon form of A.

We summarize the results of this section by the following theorem.

THEOREM 11: The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

if $\det A \neq 0$. The equation $A\mathbf{x} = \mathbf{b}$ has either no solutions, or infinitely many solutions if $\det A = 0$.

COROLLARY: The equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution (that is, a solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

with not all the x_i equal to zero) if, and only if, $\det A = 0$.

Proof: Observe that

$$\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

is always one solution of the equation $A\mathbf{x} = \mathbf{0}$. Hence, it is the only solution if $\det A \neq 0$ by Theorem 11 above. On the other hand, there exist infinitely many solutions if $\det A = 0$, and all but one of these are nontrivial. ■

EXAMPLE: For which values of λ does the equation

$$\begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

have a nontrivial solution?

Solution: We have

$$\det \begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \lambda + \lambda - 1 - \lambda = \lambda - 1$$

Hence, the equation

$$\begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

has a nontrivial solution if, and only if, $\lambda = 1$.

Appendix I

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Solution: We have

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 1 & 3 & -6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -2 & 10 & -9 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & -17 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & -17 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\left\{ \begin{array}{l} x_1 + 9x_3 = 0 \\ \\ x_2 - 5x_3 = 0 \\ \\ x_4 = 0 \end{array} \right. \qquad \Longrightarrow \qquad \left\{ \begin{array}{l} x_1 = -9x_3 \\ \\ x_2 = 5x_3 \\ \\ x_4 = 0 \end{array} \right.$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

EXAMPLE: It can be shown that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix Reduced Echelon Form

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find all solutions of the system $A\mathbf{x} = \mathbf{0}$.

Solution: Consider the system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -4x_2 - 2x_4 \\ x_3 = x_4 \\ x_5 = 0 \end{cases}$$

From this it follows that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Appendix II

EXAMPLE: Determine whether the given vectors are linearly dependent or independent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

Solution: We have

$$\begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ -1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -12 \\ 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (two) is less than the number of columns (three), the vectors are linearly dependent. Moreover, since column 1 and column 2 are the pivot columns, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for Span$\{\mathbf{v}_1, \mathbf{v}_2\}$. The dimension of this vector space is 2 (= number of pivots).

EXAMPLE: Show that the vectors (polynomials)

$$\mathbf{p}_1 = 1 + t^3, \quad \mathbf{p}_2 = 3 + t - 2t^2, \quad \mathbf{p}_3 = -t + 3t^2 - t^3$$

are linearly independent.

Solution: Consider the coefficients of each polynomial as the coordinates of a vector in $\mathbb{R}^4$. We have

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since there is a pivot in every column, the vectors are linearly independent. Moreover, since columns 1, 2, and 3 are the pivot columns, the vectors $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ form a basis for Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. The dimension of this vector space is 3 (= number of pivots) and $\mathbb{R}^3 = $ Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

EXAMPLE: Show that the vectors (polynomials)

$$\mathbf{p}_1 = 1 - 3t + 5t^2, \quad \mathbf{p}_2 = -3 + 5t - 7t^2, \quad \mathbf{p}_3 = -4 + 5t - 6t^2, \quad \mathbf{p}_4 = 1 - t^2$$

are linearly dependent.

Solution: Consider the coefficients of each polynomial as the coordinates of a vector in $\mathbb{R}^3$. We have

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 8 & 14 & -6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & 4 & 7 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (two) is less than the number of columns (four), the vectors are linearly dependent. Moreover, since columns 1 and 2 are the pivot columns, the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ form a basis for Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$. The dimension of this vector space is 2 (= number of pivots).

EXAMPLE: Show that the vectors (matrices)

$$\mathbf{m}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{m}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{m}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{m}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

are linearly dependent.

Solution: Consider the components of each matrix as the coordinates of a vector in $\mathbb{R}^4$. We have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (three) is less than the number of columns (four), the vectors are linearly dependent. Moreover, since columns 1, 2, and 3 are the pivot columns, the vectors $\mathbf{m}_1, \mathbf{m}_2$, and $\mathbf{m}_3$ form a basis for Span$\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4\}$. The dimension of this vector space is 3 (= number of pivots).

Appendix III

EXAMPLE: Find the dimension of the subspace of $\mathbb{R}^3$ which consists of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that satisfy the system of equations

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \\ 6x_1 + 6x_3 = 0 \end{cases}$$

Solution: We use row operations

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \\ 6 & 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

Since there is only one non-pivot column, the dimension of the vector space of the solutions of the system above is 1. Indeed, the reduced echelon form tells us that the original system has the same solution set as the system

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

therefore the dimension is 1.

EXAMPLE: Find the dimension of the vector space of all solutions of the system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0 \\ 3x_1 + 4x_2 + x_3 + x_4 + x_5 + 2x_6 = 0 \\ 2x_1 + 3x_2 + x_6 = 0 \end{cases}$$

Solution: We use row operations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 3 & 4 & 1 & 1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 & -2 & -1 & 0 \\ 0 & 1 & -2 & -2 & -2 & -1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 3 & 3 & 3 & 2 & 0 \\ 0 & 1 & -2 & -2 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

Since the number of non-pivot columns is 4, the dimension of the vector space of the solutions of the system above is 4. Indeed, the reduced echelon form tells us that the original system has the same solution set as the system

$$\begin{cases} x_1 + 3x_3 + 3x_4 + 3x_5 + 2x_6 = 0 \\ x_2 - 2x_3 - 2x_4 - 2x_5 - x_6 = 0 \end{cases} \implies \begin{cases} x_1 = -3x_3 - 3x_4 - 3x_5 - 2x_6 \\ x_2 = 2x_3 + 2x_4 + 2x_5 + x_6 \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3x_3 - 3x_4 - 3x_5 - 2x_6 \\ 2x_3 + 2x_4 + 2x_5 + x_6 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_1} + x_4 \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}_3} + x_6 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{u}_4}$$

One can check that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and $\mathbf{u}_4$ are linearly independent. Indeed,

$$[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4] = \begin{bmatrix} -3 & -3 & -3 & -2 \\ 2 & 2 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -3 & -3 & -2 \\ 2 & 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since each column has a pivot, vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and $\mathbf{u}_4$ are linearly independent. Therefore they form a basis for the vector space of the solutions of the given system and the dimension of this vector space is 4.