

# Linear Transformations

Recall that a system of linear equations can be rewritten in an equivalent matrix form. For example, the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 17x_4 = 18 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 = 57 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 = 19 \end{cases}$$

can be rewritten as  $A\mathbf{x} = \mathbf{b}$ , where

$$A = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 18 \\ 57 \\ 19 \end{bmatrix}$$

Indeed,

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}$$

and  $A\mathbf{x}$  is equal to  $\mathbf{b}$  if and only if the corresponding components are equal.

In the previous section we approached the problem of solving the equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

by asking whether the matrix  $A^{-1}$  exists. This approach led us to the conclusion that equation (1) has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

if

$$\det A \neq 0$$

In order to determine what happens when  $\det A = 0$ , we will approach the problem of solving (1) in an entirely different manner. We begin with the following lemma.

LEMMA 1: Let  $A$  be an  $n \times n$  matrix with elements  $a_{ij}$ , and let  $\mathbf{x}$  be a vector with components  $x_1, x_2, \dots, x_n$ . Let

$$\mathbf{a}^j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

denote the  $j$ th column of  $A$ . Then

$$A\mathbf{x} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 + \dots + x_n\mathbf{a}^n$$

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 17 \\ 46 \\ 31 \end{bmatrix} \\ &= x_1 \mathbf{a}^1 + x_2 \mathbf{a}^2 + x_3 \mathbf{a}^3 + x_4 \mathbf{a}^4 \end{aligned}$$

where

$$\mathbf{a}^1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{a}^2 = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \quad \mathbf{a}^3 = \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix}, \quad \mathbf{a}^4 = \begin{bmatrix} 17 \\ 46 \\ 31 \end{bmatrix}$$

THEOREM 9:

(a) The equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution if the columns of  $A$  are linearly independent.

(b) The equation  $A\mathbf{x} = \mathbf{b}$  has either no solution, or infinitely many solutions, if the columns of  $A$  are linearly dependent.

EXAMPLE:

(a) For which vectors

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

can we solve the equation

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{x} = \mathbf{b}?$$

Solution: The columns of the matrix  $A$  are

$$\mathbf{a}^1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}^2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}^3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

By Lemma 1,

$$A\mathbf{x} = c'_1\mathbf{a}^1 + c'_2\mathbf{a}^2 + c'_3\mathbf{a}^3$$

for some constants  $c'_1, c'_2$ , and  $c'_3$ . Since  $A\mathbf{x} = \mathbf{b}$ , it follows that this system has a solution if, and only if,

$$\mathbf{b} = c'_1\mathbf{a}^1 + c'_2\mathbf{a}^2 + c'_3\mathbf{a}^3$$

Now note that  $\mathbf{a}^1$  and  $\mathbf{a}^2$  are linearly independent while  $\mathbf{a}^3$  is the sum of  $\mathbf{a}^1$  and  $\mathbf{a}^2$ . Hence, we can solve the equation  $A\mathbf{x} = \mathbf{b}$  if, and only if,

$$\begin{aligned} \mathbf{b} &= c'_1\mathbf{a}^1 + c'_2\mathbf{a}^2 + c'_3\mathbf{a}^3 \\ &= c'_1\mathbf{a}^1 + c'_2\mathbf{a}^2 + c'_3(\mathbf{a}^1 + \mathbf{a}^2) \\ &= c'_1\mathbf{a}^1 + c'_2\mathbf{a}^2 + c'_3\mathbf{a}^1 + c'_3\mathbf{a}^2 \\ &= \underbrace{(c'_1 + c'_3)}_{c_1}\mathbf{a}^1 + \underbrace{(c'_2 + c'_3)}_{c_2}\mathbf{a}^2 \\ &= c_1\mathbf{a}^1 + c_2\mathbf{a}^2 \\ &= c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \\ 2c_1 + c_2 \end{bmatrix} \end{aligned}$$

for some constants  $c_1$  and  $c_2$ . Equivalently, one can show that  $b_3 = b_1 + b_2$ .

(b) Find all solutions of the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: Consider the corresponding system of three linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ x_1 + x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \end{cases}$$

Notice that the third equation is the sum of the first two equations. Hence, we need only consider the first two equations. The second equation says that  $x_1 = -x_3$ . Substituting this value of  $x_1$ , into the first equation gives  $x_2 = -x_3$ . Hence, all solutions of the equation  $A\mathbf{x} = \mathbf{0}$  are of the form

$$\mathbf{x} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(c) Find all solutions of the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution 1: Observe first that

$$\mathbf{x}^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is clearly one solution of this equation. Next, let  $\mathbf{x}^2$  be any other solution of this equation. We show that

$$\mathbf{x}^2 = \mathbf{x}^1 + \xi$$

where  $\xi$  is a solution of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ . Indeed, since  $\mathbf{x}^1$  and  $\mathbf{x}^2$  are solutions of  $A\mathbf{x} = \mathbf{b}$ , we get  $A\mathbf{x}^1 = \mathbf{b}$  and  $A\mathbf{x}^2 = \mathbf{b}$ , therefore

$$\mathbf{0} = \mathbf{b} - \mathbf{b} = A\mathbf{x}^2 - A\mathbf{x}^1 = A(\mathbf{x}^2 - \mathbf{x}^1)$$

This means that  $\mathbf{x}^2 - \mathbf{x}^1$  is a solution of  $A\mathbf{x} = \mathbf{0}$ . Putting  $\xi = \mathbf{x}^2 - \mathbf{x}^1$ , we get the desired result. Moreover, the sum of any solution of the nonhomogeneous equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

with a solution of the homogeneous equation is again a solution of the nonhomogeneous equation. Indeed, if  $\mathbf{x}$  is a solution of  $A\mathbf{x} = \mathbf{b}$  and  $\xi$  is a solution of  $A\mathbf{x} = \mathbf{0}$ , then  $\mathbf{x} + \xi$  is a solution of  $A\mathbf{x} = \mathbf{b}$  again, since

$$A(\mathbf{x} + \xi) = A\mathbf{x} + A\xi = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Hence, any solution of the equation  $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  is of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Solution 2: We have

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 1 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = 1 - x_3 \\ x_3 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 + (-1) \cdot x_3 \\ 1 + (-1) \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} (-1) \cdot x_3 \\ (-1) \cdot x_3 \\ 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS:

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

ECHELON FORM:

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

REDUCED ECHELON FORM:

$$\begin{bmatrix} 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

DEFINITION: A **pivot position** in a matrix is a location in  $A$  that corresponds to a leading 1 in the reduced echelon form of  $A$ . A **pivot column** is a column of  $A$  that contains a pivot position. The variables corresponding to pivot columns are called **basic variables**. The other variables are called **free variables**.

EXAMPLE: Find all solutions of the system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

Solution: We use row operations

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + 3x_2 = -5 \\ x_3 = 3 \end{cases} \implies \begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \\ x_2 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 3x_2 \\ x_2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 + (-3) \cdot x_2 \\ 0 + 1 \cdot x_2 \\ 3 + 0 \cdot x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system  $A\mathbf{x} = \mathbf{b}$ , where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Solution: We use row operations

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{cases} \implies \begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases}$$

that is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3} \cdot x_3 \\ 2 + 0 \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system  $A\mathbf{x} = \mathbf{0}$ , where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution: We use row operations

$$\begin{aligned}
 \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} &\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix} \\
 &\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix} \\
 &\sim \underbrace{\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \\
 &\sim \underbrace{\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}
 \end{aligned}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \cdot x_2 + 1 \cdot x_4 + (-3) \cdot x_5 \\ 1 \cdot x_2 + 0 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_2 + (-2) \cdot x_4 + 2 \cdot x_5 \\ 0 \cdot x_2 + 1 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_2 + 0 \cdot x_4 + 1 \cdot x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Theorem 9 is an extremely useful theorem since it establishes necessary and sufficient conditions for the equation  $A\mathbf{x} = \mathbf{b}$  to have a unique solution. However, it is often very difficult to apply Theorem 9 since it is usually quite difficult to determine whether  $n$  vectors are linearly dependent or linearly independent. Fortunately, we can relate the question of whether the columns of  $A$  are linearly dependent or linearly independent to the much simpler problem of determining whether the determinant of  $A$  is zero or nonzero. There are several different ways of accomplishing this. In this section, we will present a very elegant method which utilizes the important concept of a linear transformation.

**DEFINITION:** A **transformation** (or **function**, or **mapping**)  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a rule that assigns to each vector  $\mathbf{x}$  from  $\mathbb{R}^n$  a vector  $T(\mathbf{x})$  in  $\mathbb{R}^m$ . The set  $\mathbb{R}^n$  is called the **domain**, and  $\mathbb{R}^m$  is called the **codomain** of  $T$ .

**EXAMPLE:** Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and let

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad T(\mathbf{x}) = A\mathbf{x}$$

Find  $T(\mathbf{u})$ .

Solution: We have:

$$T(\mathbf{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

DEFINITION: A transformation  $T$  is **linear** if:

- (i)  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in the domain of  $T$
- (ii)  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all  $\mathbf{u}$  in the domain of  $T$  and all scalars  $c$

THEOREM: If  $T$  is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors  $\mathbf{u}, \mathbf{v}$  in the domain of  $T$  and all scalars  $c, d$ .

EXAMPLE: Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a transformation such that

$$T(\mathbf{x}) = A\mathbf{x}$$

where  $A$  is an  $m \times n$  matrix. Then  $T$  is linear, since

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$$

THEOREM 10: Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then there exists a unique matrix  $A$  such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact,  $A$  is the  $m \times n$  matrix whose  $j$ th column is the vector  $T(\mathbf{e}_j)$ :

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)]$$

where  $\mathbf{e}_j$  is the  $j$ th column of the identity matrix in  $\mathbb{R}^n$ , that is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE: Find a matrix  $A$  such that for any  $\mathbf{x}$  from  $\mathbb{R}^2$  we have

$$T(\mathbf{x}) = 3\mathbf{x}$$



Solution 1: We have

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

therefore

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution 2: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$T(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \quad \text{and} \quad 3\mathbf{x} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

Hence,

$$a = 3, \quad b = 0, \quad c = 0, \quad d = 3$$

so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow T(\mathbf{x}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is linear, since

$$T(\mathbf{x}) = I_n \mathbf{x}$$

where  $I_n$  is the  $n \times n$  identity matrix.

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \\ x_1 \end{bmatrix}$$

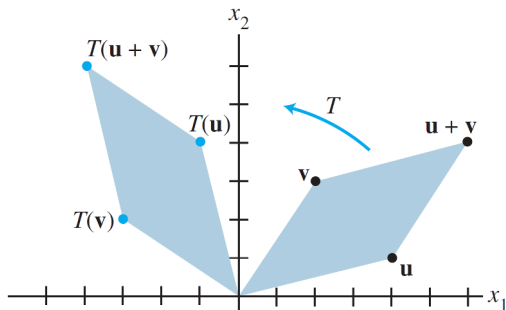
is linear, since

$$T(\mathbf{x}) = A\mathbf{x}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

EXAMPLE: Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the transformation that rotates each point in  $\mathbb{R}^2$  about the origin through an angle  $\phi$ , with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear.



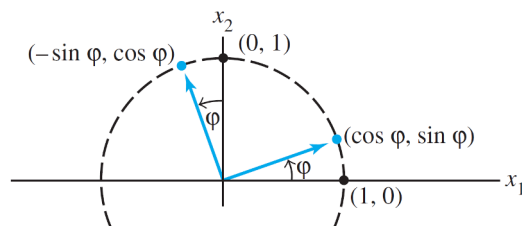
Find the standard matrix  $A$  of this transformation.

Solution: Note that

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{rotates into} \quad \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}$$

and

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{rotates into} \quad \begin{bmatrix} -\sin \phi \\ \cos \phi \end{bmatrix}$$



By Theorem 10,

$$A = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \longrightarrow \quad T(\mathbf{x}) = \begin{bmatrix} 1 \\ x_1^2 + x_2^2 \end{bmatrix}$$

is *not* linear. Indeed, since

$$T(2\mathbf{x}) = \begin{bmatrix} 1 \\ (2x_1)^2 + (2x_2)^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4x_1^2 + 4x_2^2 \end{bmatrix}$$

and

$$2T(\mathbf{x}) = \begin{bmatrix} 2 \\ 2x_1^2 + 2x_2^2 \end{bmatrix}$$

Since

$$T(2\mathbf{x}) \neq 2T(\mathbf{x})$$

it follows that  $T$  is *not* linear.

DEFINITION: A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be **invertible** if there exists a function  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

We call  $S$  the **inverse** of  $T$  and write it as  $T^{-1}$ .

LEMMA 2: Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation and let  $A$  be the standard matrix for  $T$ . Then  $T$  is invertible if and only if  $A$  is an invertible matrix. In that case, the linear transformation  $S$  given by

$$S(\mathbf{x}) = A^{-1}\mathbf{x}$$

is the unique function satisfying both equations

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

We are now ready to relate the problem of determining whether the columns of an  $n \times n$  matrix  $A$  are linearly dependent or linearly independent to the much simpler problem of determining whether the determinant of  $A$  is zero or nonzero.

LEMMA 3: The columns of an  $n \times n$  matrix  $A$  are linearly independent if, and only if,  $\det A \neq 0$ .

Proof: We prove Lemma 3 by the following complex, but very clever argument.

(1) The columns of  $A$  are linearly independent if, and only if, the equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x}$  for every  $\mathbf{b}$  in  $\mathbb{R}^n$ . This statement is just a reformulation of Theorem 9.

(2) The equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x}$  for every  $\mathbf{b}$  if, and only if, the linear transformation  $T(\mathbf{x}) = A\mathbf{x}$  has an inverse.

(3) From Lemma 2, the linear transformation  $T$  has an inverse if, and only if, the matrix  $A^{-1}$  exists.

(4) Finally, the matrix  $A^{-1}$  exists if, and only if,  $\det A \neq 0$ . This is the content of Theorem 8, Section 3.6. Therefore, we conclude that the columns of  $A$  are linearly independent if, and only if,  $\det A \neq 0$ . ■

EXAMPLE: Show that vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent. Then show that vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent.

Solution: We have

$$\det[\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Since the determinant is equal to zero, the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  are linearly dependent by Lemma 3 above. Similarly,

$$\det[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Since the determinant is not equal to zero, the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent by Lemma 3 above.

REMARK: One can show that the following stronger statement is true: Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_m$  from  $\mathbb{R}^n$  are linearly independent if, and only if, an echelon form of the matrix

$$A = [\mathbf{v}_1 \dots \mathbf{v}_m]$$

has a pivot in every column; these vectors are dependent otherwise. Moreover, the pivot columns of the matrix  $A$  form a basis for the vector space spanned by the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_m$ .

EXAMPLE: Let  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  and  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  be the same as above. We have

$$A = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (two) is lesser than the number of columns (three), the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  are linearly dependent. Moreover, the vectors  $\mathbf{u}_1, \mathbf{u}_2$  (the pivot columns of the matrix  $A$ ) form a basis for the vector space spanned by the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ . So,

$$\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$$

Furthermore, by Theorem 2 from Section 3.3, the dimension of  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  (and therefore the dimension of  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ ) is *two*, since  $\mathbf{u}_1, \mathbf{u}_2$  are *two* linearly independent vectors.

Similarly,

$$B = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (three) is equal to the number of columns (three), the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent. Moreover, the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  (the pivot columns of the matrix  $B$ ) form a basis for the vector space spanned by the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . Furthermore, by Theorem 3 from Section 3.3, these vectors form a basis for  $\mathbb{R}^3$  (and therefore  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$ ), since these *three* vectors are linearly independent and the dimension of  $\mathbb{R}^3$  is also *three*.

We summarize the results of this section by the following theorem.

THEOREM 11: The equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

if  $\det A \neq 0$ . The equation  $A\mathbf{x} = \mathbf{b}$  has either no solutions, or infinitely many solutions if  $\det A = 0$ .

COROLLARY: The equation  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution (that is, a solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

with not all the  $x_i$  equal to zero) if, and only if,  $\det A = 0$ .

Proof: Observe that

$$\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

is always one solution of the equation  $A\mathbf{x} = \mathbf{0}$ . Hence, it is the only solution if  $\det A \neq 0$  by Theorem 11 above. On the other hand, there exist infinitely many solutions if  $\det A = 0$ , and all but one of these are nontrivial. ■

EXAMPLE: For which values of  $\lambda$  does the equation

$$\begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

have a nontrivial solution?

Solution: We have

$$\det \begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \lambda + \lambda - 1 - \lambda = \lambda - 1$$

Hence, the equation

$$\begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

has a nontrivial solution if, and only if,  $\lambda = 1$ .

## Appendix I

EXAMPLE: Find all solutions of the system  $A\mathbf{x} = \mathbf{0}$ , where

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Solution: We have

$$\begin{aligned} \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{bmatrix} &\sim \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 1 & 3 & -6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -2 & 10 & -9 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & -17 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \\ &\sim \underbrace{\begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & -17 & 0 \end{bmatrix}}_{\text{Echelon Form}} \\ &\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & -6 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ &\sim \underbrace{\begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}} \end{aligned}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + 9x_3 = 0 \\ x_2 - 5x_3 = 0 \\ x_4 = 0 \end{cases} \implies \begin{cases} x_1 = -9x_3 \\ x_2 = 5x_3 \\ x_4 = 0 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

EXAMPLE: It can be shown that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix Reduced Echelon Form

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find all solutions of the system  $A\mathbf{x} = \mathbf{0}$ .

Solution: Consider the system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0 \end{cases} \implies \begin{cases} x_1 = -4x_2 - 2x_4 \\ x_3 = x_4 \\ x_5 = 0 \end{cases}$$

From this it follows that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix}$$

- Find a basis and the dimension of  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ .
- Find all solutions of the system  $A\mathbf{x} = \mathbf{0}$ , where  $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4 \ \mathbf{v}_5]$ .
- Find a basis and the dimension of the vector space of the solutions of the system  $A\mathbf{x} = \mathbf{0}$ .

Solution:

(a) We have

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since column 1 and column 2 are the pivot columns, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

form a basis for  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ . The dimension of this vector space is 2.

(b) We have

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 & 0 \\ 2 & 1 & 1 & -1 & 0 & 0 \\ 0 & 9 & -3 & -1 & -6 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 2 & 0 & 3 & 0 \\ 0 & 9 & -3 & -1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases} \implies \begin{cases} x_1 = -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ x_2 = \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \end{cases}$$

therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{u}_3}$$

(c) One can check that the vectors  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ , and  $\mathbf{u}_3$  are linearly independent. Indeed,

$$[\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{bmatrix} -\frac{2}{3} & \frac{4}{9} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{9} & \frac{2}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{2}{3} & \frac{4}{9} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{9} & \frac{2}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since each column has a pivot, vectors  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ , and  $\mathbf{u}_3$  are linearly independent. Therefore they form a basis for the vector space of the solutions of the system  $A\mathbf{x} = \mathbf{0}$ . The dimension of this vector space is 3.



## Appendix II

EXAMPLE: Determine whether the given vectors are linearly dependent or independent.

$$\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

Solution: We have

$$\begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ -1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -12 \\ 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Since the number of pivots (two) is less than the number of columns (three), the vectors are linearly dependent.

EXAMPLE: What is the dimension of the subspace of  $\mathbb{R}^3$  which consists of all vectors  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  that

satisfy the system of equations

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \\ 6x_1 + 6x_3 = 0 \end{cases}$$

Solution: We use row operations

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \\ 6 & 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

therefore the dimension is 1.