Linear Transformations

Recall that a system of linear equations can be rewritten in an equivalent matrix form. For example, the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 17x_4 = 18 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 = 57 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 = 19 \end{cases}$$

can be rewritten as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 18 \\ 57 \\ 19 \end{bmatrix}$$

Indeed,

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}$$

and $A\mathbf{x}$ is equal to $\mathbf{b}$ if and only if the corresponding components are equal.

In the previous section we approached the problem of solving the equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

by asking whether the matrix A^{-1} exists. This approach led us to the conclusion that equation (1) has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

if

$$\det A \neq 0$$

In order to determine what happens when $\det A = 0$, we will approach the problem of solving (1) in an entirely different manner. We begin with the following lemma.

LEMMA 1: Let A be an $n \times n$ matrix with elements a_{ij}, and let $\mathbf{x}$ be a vector with components $x_1,\ x_2, \dots, x_n$. Let

$$\mathbf{a^j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

denote the jth column of A. Then

$$A\mathbf{x} = x_1\mathbf{a^1} + x_2\mathbf{a^2} + \dots + x_n\mathbf{a^n}$$

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 17 \\ 46 \\ 31 \end{bmatrix}$$

$$= x_1\mathbf{a^1} + x_2\mathbf{a^2} + x_3\mathbf{a^3} + x_4\mathbf{a^4}$$

where

$$\mathbf{a^1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{a^2} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \quad \mathbf{a^3} = \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix}, \quad \mathbf{a^4} = \begin{bmatrix} 17 \\ 46 \\ 31 \end{bmatrix}$$

THEOREM 9:

(a) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution if the columns of A are linearly independent.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has either no solution, or infinitely many solutions, if the columns of A are linearly dependent.

EXAMPLE:

(a) For which vectors

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

can we solve the equation

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{x} = \mathbf{b}?$$

Solution: The columns of the matrix A are

$$\mathbf{a^1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a^2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a^3} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

By Lemma 1,

$$A\mathbf{x} = c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^3}$$

for some constants c_1', c_2', and c_3'. Since $A\mathbf{x} = \mathbf{b}$, it follows that this system has a solution if, and only if,

$$\mathbf{b} = c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^3}$$

Now note that $\mathbf{a^1}$ and $\mathbf{a^2}$ are linearly independent while $\mathbf{a^3}$ is the sum of $\mathbf{a^1}$ and $\mathbf{a^2}$. Hence, we can solve the equation $A\mathbf{x} = \mathbf{b}$ if, and only if,

$$\begin{aligned}
\mathbf{b} &= c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^3} \\
&= c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'(\mathbf{a^1} + \mathbf{a^2}) \\
&= c_1'\mathbf{a^1} + c_2'\mathbf{a^2} + c_3'\mathbf{a^1} + c_3'\mathbf{a^2} \\
&= \underbrace{(c_1' + c_3')}_{c_1}\mathbf{a^1} + \underbrace{(c_2' + c_3')}_{c_2}\mathbf{a^2} \\
&= c_1\mathbf{a^1} + c_2\mathbf{a^2} \\
&= c_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \\ 2c_1 + c_2 \end{bmatrix}
\end{aligned}$$

for some constants c_1 and c_2. Equivalently, one can show that $b_3 = b_1 + b_2$.

(b) Find all solutions of the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: Consider the corresponding system of three linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 &= 0 \\ x_1 \qquad\;\; + x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{cases}$$

Notice that the third equation is the sum of the first two equations. Hence, we need only consider the first two equations. The second equation says that $x_1 = -x_3$. Substituting this value of x_1, into the first equation gives $x_2 = -x_3$. Hence, all solutions of the equation $A\mathbf{x} = \mathbf{0}$ are of the form

$$\mathbf{x} = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(c) Find all solutions of the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution 1: Observe first that

$$\mathbf{x^1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is clearly one solution of this equation. Next, let $\mathbf{x^2}$ be any other solution of this equation. We show that

$$\mathbf{x^2} = \mathbf{x^1} + \xi$$

where ξ is a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Indeed, since $\mathbf{x^1}$ and $\mathbf{x^2}$ are solutions of $A\mathbf{x} = \mathbf{b}$, we get $A\mathbf{x^1} = \mathbf{b}$ and $A\mathbf{x^2} = \mathbf{b}$, therefore

$$\mathbf{0} = \mathbf{b} - \mathbf{b} = A\mathbf{x^2} - A\mathbf{x^1} = A(\mathbf{x^2} - \mathbf{x^1})$$

This means that $\mathbf{x^2} - \mathbf{x^1}$ is a solution of $A\mathbf{x} = \mathbf{0}$. Putting $\xi = \mathbf{x^2} - \mathbf{x^1}$, we get the desired result. Moreover, the sum of any solution of the nonhomogeneous equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

with a solution of the homogeneous equation is again a solution of the nonhomogeneous equation. Indeed, if $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$ and ξ is a solution of $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} + \xi$ is a solution of $A\mathbf{x} = \mathbf{b}$ again, since

$$A(\mathbf{x} + \xi) = A\mathbf{x} + A\xi = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Hence, any solution of the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Solution 2: We have

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this it follows that the original system has the same solution set as the system

$$\left\{ \begin{array}{ccccc} x_1 & & + & x_3 & = 0 \\ & x_2 & + & x_3 & = 1 \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{l} x_1 = -x_3 \\ x_2 = 1 - x_3 \\ x_3 \text{ is free} \end{array} \right.$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 + (-1) \cdot x_3 \\ 1 + (-1) \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} (-1) \cdot x_3 \\ (-1) \cdot x_3 \\ 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS:

1. Add a multiple of one row to another.

2. Multiply a row by a nonzero constant.

3. Interchange two rows.

ECHELON FORM:

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

REDUCED ECHELON FORM:

$$\begin{bmatrix} 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

DEFINITION: A **pivot position** in a matrix is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position. The variables corresponding to pivot columns are called **basic variables**. The other variables are called **free variables**.

EXAMPLE: Find all solutions of the system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

Solution: We use row operations

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + 3x_2 = -5 \\ x_3 = 3 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \\ x_2 \text{ is free} \end{cases}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 3x_2 \\ x_2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 + (-3) \cdot x_2 \\ 0 + 1 \cdot x_2 \\ 3 + 0 \cdot x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Solution: We use row operations

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases}$$

that is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3} \cdot x_3 \\ 2 + 0 \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution: We use row operations

$$\left[\begin{array}{rrrrrr} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\sim \underbrace{\left[\begin{array}{rrrrrr} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Echelon Form}}$$

$$\sim \underbrace{\left[\begin{array}{rrrrrr} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \cdot x_2 + 1 \cdot x_4 + (-3) \cdot x_5 \\ 1 \cdot x_2 + 0 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_2 + (-2) \cdot x_4 + 2 \cdot x_5 \\ 0 \cdot x_2 + 1 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_2 + 0 \cdot x_4 + 1 \cdot x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Theorem 9 is an extremely useful theorem since it establishes necessary and sufficient conditions for the equation $A\mathbf{x} = \mathbf{b}$ to have a unique solution. However, it is often very difficult to apply Theorem 9 since it is usually quite difficult to determine whether n vectors are linearly dependent or linearly independent. Fortunately, we can relate the question of whether the columns of A are linearly dependent or linearly independent to the much simpler problem of determining whether the determinant of A is zero or nonzero. There are several different ways of accomplishing this. In this section, we will present a very elegant method which utilizes the important concept of a linear transformation.

DEFINITION: A **transformation** (or **function**, or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ from $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain**, and $\mathbb{R}^m$ is called the **codomain** of T.

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and let

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T(\mathbf{x}) = A\mathbf{x}$$

Find $T(\mathbf{u})$.

Solution: We have:

$$T(\mathbf{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

DEFINITION: A transformation T is **linear** if:

$$(i) \quad T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \text{ in the domain of } T$$

$$(ii) \quad T(c\mathbf{u}) = cT(\mathbf{u}) \quad \text{for all } \mathbf{u} \text{ in the domain of } T \text{ and all scalars } c$$

THEOREM: If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

EXAMPLE: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation such that

$$T(\mathbf{x}) = A\mathbf{x}$$

where A is an $m \times n$ matrix. Then T is linear, since

$$T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$$

THEOREM 10: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e_j})$:

$$A = [T(\mathbf{e_1}), T(\mathbf{e_2}), \ldots, T(\mathbf{e_n})]$$

where $\mathbf{e_j}$ is the jth column of the identity matrix in $\mathbb{R}^n$, that is,

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE: Find a matrix A such that for any $\mathbf{x}$ from $\mathbb{R}^2$ we have

$$T(\mathbf{x}) = 3\mathbf{x}$$

Solution 1: We have

$$T(\mathbf{e_1}) = 3\mathbf{e_1} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e_2}) = 3\mathbf{e_2} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

therefore

$$A = [T(\mathbf{e_1}), T(\mathbf{e_2})] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution 2: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$T(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \qquad \text{and} \qquad 3\mathbf{x} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

Hence,

$$a = 3,\ b = 0,\ c = 0,\ d = 3$$

so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow T(\mathbf{x}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is linear, since

$$T(\mathbf{x}) = I_n\mathbf{x}$$

where I_n is the $n \times n$ identity matrix.

EXAMPLE: The transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \\ x_1 \end{bmatrix}$$

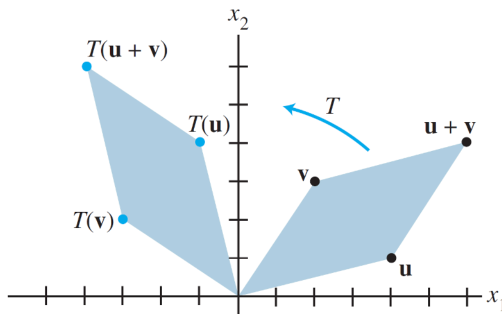
is linear, since

$$T(\mathbf{x}) = A\mathbf{x}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

EXAMPLE: Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ϕ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear.



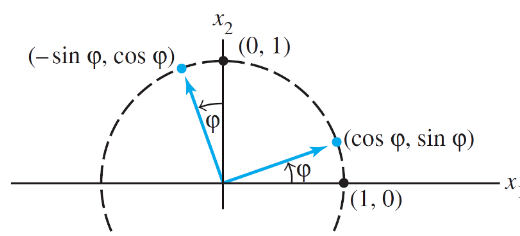
Find the standard matrix A of this transformation.

Solution: Note that

$$\mathbf{e_1}=\begin{bmatrix}1\\0\end{bmatrix}\quad\text{rotates into}\quad\begin{bmatrix}\cos\phi\\\sin\phi\end{bmatrix}$$

and

$$\mathbf{e_2}=\begin{bmatrix}0\\1\end{bmatrix}\quad\text{rotates into}\quad\begin{bmatrix}-\sin\phi\\\cos\phi\end{bmatrix}$$



By Theorem 10,

$$A=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}$$

EXAMPLE: The transformation

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\quad\longrightarrow\quad T(\mathbf{x})=\begin{bmatrix}1\\x_1^2+x_2^2\end{bmatrix}$$

is *not* linear. Indeed, since

$$T(2\mathbf{x})=\begin{bmatrix}1\\(2x_1)^2+(2x_2)^2\end{bmatrix}=\begin{bmatrix}1\\4x_1^2+4x_2^2\end{bmatrix}$$

and

$$2T(\mathbf{x})=\begin{bmatrix}2\\2x_1^2+2x_2^2\end{bmatrix}$$

Since

$$T(2\mathbf{x})\neq 2T(\mathbf{x})$$

it follows that T is *not* linear.

DEFINITION: A linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$ is said to be **invertible** if there exists a function $S:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$S(T(\mathbf{x}))=\mathbf{x}\quad\text{for all }\mathbf{x}\text{ in }\mathbb{R}^n$$
$$T(S(\mathbf{x}))=\mathbf{x}\quad\text{for all }\mathbf{x}\text{ in }\mathbb{R}^n$$

We call S the **inverse** of T and write it as T^{-1}.

LEMMA 2: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by

$$S(\mathbf{x}) = A^{-1}\mathbf{x}$$

is the unique function satisfying both equations

$$S(T(\mathbf{x})) = \mathbf{x} \qquad \text{and} \qquad T(S(\mathbf{x})) = \mathbf{x}$$

We are now ready to relate the problem of determining whether the columns of an $n \times n$ matrix A are linearly dependent or linearly independent to the much simpler problem of determining whether the determinant of A is zero or nonzero.

LEMMA 3: The columns of an $n \times n$ matrix A are linearly independent if, and only if, $\det A \neq 0$.

Proof: We prove Lemma 3 by the following complex, but very clever argument.

(1) The columns of A are linearly independent if, and only if, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every $\mathbf{b}$ in $\mathbb{R}^n$. This statement is just a reformulation of Theorem 9.

(2) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every $\mathbf{b}$ if, and only if, the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ has an inverse.

(3) From Lemma 2, the linear transformation T has an inverse if, and only if, the matrix A^{-1} exists.

(4) Finally, the matrix A^{-1} exists if, and only if, $\det A \neq 0$. This is the content of Theorem 8, Section 3.6. Therefore, we conclude that the columns of A are linearly independent if, and only if, $\det A \neq 0$.

REMARK: One can show that the following stronger statement is true: The vectors $\mathbf{v_1}, \ldots, \mathbf{v_m}$ from $\mathbb{R}^n$ are linearly independent if, and only if, an echelon form of the matrix $[\mathbf{v_1}, \ldots, \mathbf{v_m}]$ has a pivot in every column; these vectors are dependent otherwise.

We summarize the results of this section by the following theorem.

THEOREM 11: The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ if $\det A \neq 0$. The equation $A\mathbf{x} = \mathbf{b}$ has either no solutions, or infinitely many solutions if $\det A = 0$.

COROLLARY: The equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution (that is, a solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

with not all the x_i equal to zero) if, and only if, $\det A = 0$.

Solutions: Observe that

$$\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

is always one solution of the equation $A\mathbf{x} = \mathbf{0}$. Hence, it is the only solution if $\det A \neq 0$. On the other hand, there exist infinitely many solutions if $\det A = 0$, and all but one of these are nontrivial.

EXAMPLE: For which values of λ does the equation

$$\begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

have a nontrivial solution?

Solution: We have

$$\det \begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \lambda + \lambda - 1 - \lambda = \lambda - 1$$

Hence, the equation

$$\begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

has a nontrivial solution if, and only if, $\lambda = 1$.

Appendix

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Solution: We have

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 1 & 3 & -6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -2 & 10 & -9 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & -17 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & -17 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + 9x_3 = 0 \\ x_2 - 5x_3 = 0 \\ x_4 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -9x_3 \\ x_2 = 5x_3 \\ x_4 = 0 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

EXAMPLE: It can be shown that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix Reduced Echelon Form

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find all solutions of the system $A\mathbf{x} = \mathbf{0}$.

Solution: Consider the system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -4x_2 - 2x_4 \\ x_3 = x_4 \\ x_5 = 0 \end{cases}$$

From this it follows that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE: Find all solutions of the system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

Solution: We have

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the original system has the same solution set as the system

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 = -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ x_2 = \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \end{cases}$$

therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$