Solutions of Simultaneous Linear Equations

DEFINITION: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b_1}, \ldots, \mathbf{b_p}$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b_1}, \ldots, A\mathbf{b_p}$. That is,

$$AB = A[\mathbf{b_1} \ \ \mathbf{b_2} \ \ldots \ \mathbf{b_p}] = [A\mathbf{b_1} \ \ A\mathbf{b_2} \ \ldots \ A\mathbf{b_p}]$$

ROW-COLUMN RULE FOR COMPUTING AB: If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2 \cdot 4 + 3 \cdot 1 & 2 \cdot 3 + 3 \cdot (-2) & 2 \cdot 6 + 3 \cdot 3 \\ 1 \cdot 4 + (-5) \cdot 1 & 1 \cdot 3 + (-5) \cdot (-2) & 1 \cdot 6 + (-5) \cdot 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Note that BA is undefined.

EXAMPLE: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}$$

If possible, compute:

(a) AB

(b) $AC + B^2$

(c) $AB + C^2$

Solution: We have:

(a) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$

(b) Impossible.

(c) We have

$$AB + C^2 = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -16 \\ 16 & 21 \end{bmatrix} = \begin{bmatrix} 19 & -8 \\ 32 & 30 \end{bmatrix}$$

PROPERTIES: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(B + C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$

THEOREM: Let A and B be square matrices. Then

(a) $\det(AB) = \det A \det B$

(b) $(AB)^T = B^T A^T$

WARNING

(I) In general, $AB \neq BA$.

EXAMPLE:

1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So,

$$AB \neq BA$$

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = [32]$$

and

$$BA = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

(II) If $AB = AC$, then it is not true in general that $B = C$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = AC, \quad \text{but} \quad B \neq C$$

(III) If $AB = 0$, then it is not true in general that $A = 0$ or $B = 0$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = 0, \quad \text{but} \quad A \neq 0 \quad \text{and} \quad B \neq 0$$

DEFINITION: The **identity matrix** I is the $n \times n$ matrix of the form

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

MAIN PROPERTY:

$$AI = IA = A$$

DEFINITION: An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

In this case, C is an **inverse** of A and is denoted by A^{-1}. So,

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

EXAMPLE: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. In fact, we have

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

THEOREM: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

EXAMPLE: Solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + 4x_2 = 6 \end{cases}$$

Solution: Let

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

then the system above can be rewritten as

$$A\mathbf{x} = B$$

Multiplying both sides by A^{-1} gives

$$A^{-1}(A\mathbf{x}) = A^{-1}B$$

But $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}$, therefore

$$\mathbf{x} = A^{-1}B$$

hence

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $x_1 = 2$ and $x_2 = 1$.

THEOREM: An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

PROPERTIES: Let A and B be invertible $n \times n$ matrices. Then

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(A^T)^{-1} = (A^{-1})^T$

ELEMENTARY ROW OPERATIONS:

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

ALGORITHM FOR FINDING A^{-1}:

1. Row reduce the augmented matrix $[A\ I]$.
2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.
3. Otherwise, A does not have an inverse.

EXAMPLE: Let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find A^{-1}.

Solution: We have

$$\left[\begin{array}{rrr|rrr} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

therefore

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

FORMULA FOR A^{-1}

DEFINITION: For any $n \times n$ matrix A, let A_{ij} be the **submatrix** of A, formed by deleting row i and column j.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Then

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

THEOREM (AN INVERSE FORMULA): Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}^T$$

where

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Find A^{-1}.

Solution 1:

Step 1: One can verify that $\det A = 27$.

Step 2: We have

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\det A_{11} = -48 \quad \det A_{12} = -42 \quad \det A_{13} = -3$$

$$C_{11} = -48 \qquad C_{12} = 42 \qquad C_{13} = -3$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$\det A_{21} = -24 \quad \det A_{22} = -21 \quad \det A_{23} = -6$$

$$C_{21} = 24 \qquad C_{22} = -21 \qquad C_{23} = 6$$

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\det A_{31} = -3 \quad \det A_{32} = -6 \quad \det A_{33} = -3$$

$$C_{31} = -3 \qquad C_{32} = 6 \qquad C_{33} = -3$$

Step 3:

$$A^{-1} = \frac{1}{27}\begin{bmatrix} -48 & 42 & -3 \\ 24 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix}^T = \frac{1}{27}\begin{bmatrix} -48 & 24 & -3 \\ 42 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix} = \begin{bmatrix} -16/9 & 8/9 & -1/9 \\ 14/9 & -7/9 & 2/9 \\ -1/9 & 2/9 & -1/9 \end{bmatrix}$$

Solution 2: We have

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -21 & -7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & -9 & 1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16/9 & 8/9 & -1/9 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array}\right]$$

and the same result follows.

THEOREM (Cramer's Rule): Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in R^n, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n$$

EXAMPLE: Solve using Cramer's rule

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 4x_2 = -7 \end{cases}$$

Solution: We have

$$x_1 = \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{4 - 14}{4 - (-6)} = \frac{-10}{10} = -1$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{-7 - 3}{10} = \frac{-10}{10} = -1$$