

Solutions of Simultaneous Linear Equations

DEFINITION: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \dots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \dots, A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$$

ROW-COLUMN RULE FOR COMPUTING AB : If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B . If $(AB)_{ij}$ denotes the (i, j) -entry in AB , and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2 \cdot 4 + 3 \cdot 1 & 2 \cdot 3 + 3 \cdot (-2) & 2 \cdot 6 + 3 \cdot 3 \\ 1 \cdot 4 + (-5) \cdot 1 & 1 \cdot 3 + (-5) \cdot (-2) & 1 \cdot 6 + (-5) \cdot 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Note that BA is undefined.

EXAMPLE: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}$$

If possible, compute:

(a) AB

(b) $AC + B^2$

(c) $AB + C^2$

Solution: We have:

$$(a) \ AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 \\ 2 \cdot 1 + 1 \cdot 2 + 4 \cdot 3 & 2 \cdot 0 + 1 \cdot 1 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$$

(b) Impossible.

(c) We have

$$\begin{aligned} AB + C^2 &= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 \cdot 3 + (-2) \cdot 2 & 3 \cdot (-2) + (-2) \cdot 5 \\ 2 \cdot 3 + 5 \cdot 2 & 2 \cdot (-2) + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -16 \\ 16 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 19 & -8 \\ 32 & 30 \end{bmatrix} \end{aligned}$$

PROPERTIES: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(B + C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$

THEOREM: Let A and B be square matrices. Then

(a) $\det(AB) = \det A \det B$

(b) $(AB)^T = B^T A^T$

WARNING

(I) In general, $AB \neq BA$.

EXAMPLE:

1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So,

$$AB \neq BA$$

2. Let $A = [1 \ 2 \ 3]$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then

$$AB = [1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6] = [32]$$

and

$$BA = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

(II) If $AB = AC$, then it is not true in general that $B = C$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = AC, \quad \text{but} \quad B \neq C$$

(III) If $AB = 0$, then it is not true in general that $A = 0$ or $B = 0$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = 0, \quad \text{but} \quad A \neq 0 \quad \text{and} \quad B \neq 0$$

DEFINITION: The **identity matrix** I is the $n \times n$ matrix of the form

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

MAIN PROPERTY:

$$AI = IA = A$$

DEFINITION: An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

In this case, C is an **inverse** of A and is denoted by A^{-1} . So,

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

EXAMPLE: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. In fact, we have

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

THEOREM: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

EXAMPLE: Solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + 4x_2 = 6 \end{cases}$$

Solution: Let

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

then the system above can be rewritten as

$$A\mathbf{x} = B$$

Multiplying both sides by A^{-1} gives

$$A^{-1}(A\mathbf{x}) = A^{-1}B$$

But $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}$, therefore

$$\mathbf{x} = A^{-1}B$$

hence

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 12 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

so $x_1 = 2$ and $x_2 = 1$.

THEOREM: An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

PROPERTIES: Let A and B be invertible $n \times n$ matrices. Then

- (a) $(A^{-1})^{-1} = A$
- (b) $(AB)^{-1} = B^{-1}A^{-1}$
- (c) $(A^T)^{-1} = (A^{-1})^T$
- (d) $\det(A^{-1}) = (\det A)^{-1}$

ELEMENTARY ROW OPERATIONS:

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

ALGORITHM FOR FINDING A^{-1} :

1. Row reduce the augmented matrix $[A \ I]$.
2. If A is row equivalent to I , then $[A \ I]$ is row equivalent to $[I \ A^{-1}]$.
3. Otherwise, A does not have an inverse.

EXAMPLE: Let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find A^{-1} .

Solution: We have

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] &\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array} \right] \end{aligned}$$

therefore

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

FORMULA FOR A^{-1}

DEFINITION: For any $n \times n$ matrix A , let A_{ij} be the **submatrix** of A , formed by deleting row i and column j .

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Then

$$\begin{aligned} A_{11} &= \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} & A_{12} &= \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} & A_{13} &= \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} & A_{22} &= \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} & A_{23} &= \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \\ A_{31} &= \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} & A_{32} &= \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} & A_{33} &= \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \end{aligned}$$

THEOREM (AN INVERSE FORMULA): Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

where

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Find A^{-1} .

Solution 1:

Step 1: One can verify that $\det A = 27$.

Step 2: We have

$$\begin{aligned} A_{11} &= \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} & A_{12} &= \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} & A_{13} &= \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \\ \det A_{11} &= -48 & \det A_{12} &= -42 & \det A_{13} &= -3 \\ C_{11} &= -48 & C_{12} &= 42 & C_{13} &= -3 \end{aligned}$$

$$\begin{aligned} A_{21} &= \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} & A_{22} &= \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} & A_{23} &= \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \\ \det A_{21} &= -24 & \det A_{22} &= -21 & \det A_{23} &= -6 \\ C_{21} &= 24 & C_{22} &= -21 & C_{23} &= 6 \end{aligned}$$

$$\begin{aligned} A_{31} &= \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} & A_{32} &= \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} & A_{33} &= \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \\ \det A_{31} &= -3 & \det A_{32} &= -6 & \det A_{33} &= -3 \\ C_{31} &= -3 & C_{32} &= 6 & C_{33} &= -3 \end{aligned}$$

Step 3:

$$A^{-1} = \frac{1}{27} \begin{bmatrix} -48 & 42 & -3 \\ 24 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix}^T = \frac{1}{27} \begin{bmatrix} -48 & 24 & -3 \\ 42 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix} = \begin{bmatrix} -16/9 & 8/9 & -1/9 \\ 14/9 & -7/9 & 2/9 \\ -1/9 & 2/9 & -1/9 \end{bmatrix}$$

Solution 2: We have

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 0 & 0 & 0 & 1 \end{array} \right] &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -21 & -7 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & -9 & 1 & -2 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16/9 & 8/9 & -1/9 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array} \right] \end{aligned}$$

and the same result follows.

THEOREM (Cramer's Rule): Let A be an invertible $n \times n$ matrix. For any \mathbf{b} in \mathbb{R}^n , the unique solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots, n$$

EXAMPLE: Solve using Cramer's rule

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 4x_2 = -7 \end{cases}$$

Solution: We have

$$\begin{aligned} x_1 &= \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{4 - 14}{4 - (-6)} = \frac{-10}{10} = -1 \\ x_2 &= \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{-7 - 3}{10} = \frac{-10}{10} = -1 \end{aligned}$$