The Theory of Determinants

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

DEFINITION: The **determinant** of an $n \times n$ matrix A is the following sum:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \ldots + (-1)^{n+1} a_{1n} \det A_{1n}$$

where A_{1j} are submatrices formed by deleting from A the first row and jth column.

EXAMPLE: Find $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$.

Solution: We have

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

Since

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 2(3-0) + (-1)(0-3) = 9$$

and

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = -2(-1) = 2$$

it follows that the determinant is equal to $9 - 2 = 7$.

REMARK: One can find

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

in an easier way expanding by row two:

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3(2-(-1)) = 9$$

THEOREM: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

EXAMPLE:

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 24$$

THEOREM: We have $\det A = 0$

(a) if A contains a zero-row or zero-column.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(b) if A contains two equal rows or columns.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$

(c) if some row (column) of A is a multiple of some other row (column) of A.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 3 & 5 & 7 \end{vmatrix} = 0$

THEOREM: Let A be a square matrix.

(a) If a multiple of one row (column) of A is added to another row (column) to produce a matrix B, then

$$\det A = \det B$$

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2$

(b) If two rows (columns) of A are interchanged to produce B, then

$$\det A = -\det B$$

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = -\begin{vmatrix} 3 & 3 & 8 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 8 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

(c) If one row (column) of A is multiplied by k to produce B, then

$$\det B = k \det A$$

EXAMPLE: $100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 100 & 300 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 100 & 200 \end{vmatrix} = \begin{vmatrix} 100 & 3 \\ 100 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 300 \\ 1 & 200 \end{vmatrix} = \begin{vmatrix} 10 & 30 \\ 10 & 20 \end{vmatrix}$

REMARK: Note that $\begin{vmatrix} 10 & 30 \\ 10 & 20 \end{vmatrix} = 100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$, but $\begin{bmatrix} 10 & 30 \\ 10 & 20 \end{bmatrix} = 10 \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$.

EXAMPLE: Find

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix}$$

Solution: We have

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & -9 & -9 & -9 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix} = (-9)\begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix}$$

$$= (-9)\begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

Since the last two rows are equal, the determinant is equal to 0.

DEFINITION: Let A be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix} \qquad B^T = \begin{bmatrix} -3 & 4 & 8 \\ 1 & 7 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

PROPERTIES: Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$ for any scalar r

THEOREM: Let A be a square matrix. Then

$$\det A^T = \det A$$

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 4z = 7 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

Solution: From Equation 3, you know the value z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$y + 4(2) = 7 \qquad \text{Substitute 2 for } z.$$

$$y = -1. \qquad \text{Solve for } y.$$

Finally, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \qquad \text{Substitute } -1 \text{ for } y \text{ and 2 for } z.$$

$$x = 1. \qquad \text{Solve for } x.$$

The solution is $x = 1$, $y = -1$ and $z = 2$.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

Solution: Because the leading coefficient of the first equation is 1, you can begin by saving the x at the upper left and eliminating the other x-terms from the first column.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that all but the first x have been eliminated from the first column, go to work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{3}$ produces a new third equation.

This is the same system that was solved in the previous Example. As in that example, you can conclude that the solution is $x = 1$, $y = -1$, and $z = 2$.

The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the augmented matrix of the system, by writing only the coefficients and constants that appear in the equations. Here is an example.

Linear system

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

The next Example demonstrates the elementary row operations described above.

EXAMPLE:

(a) Add -2 times the first row of the original matrix to the third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} \\ \\ -2R_1 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

(b) Multiply the first row of the original matrix by $\frac{1}{2}$.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} \frac{1}{2}R_1 \rightarrow \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

(c) Interchange the first and second rows of the original matrix.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} R_2 \\ R_1 \\ \\ \end{matrix} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

Symbol	Description
$R_i + kR_j \rightarrow R_i$	Change the ith row by adding k times row j to it, then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \leftrightarrow R_j$	Interchange the ith and jth rows.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

Solution: Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison, we write both the system of equations and its augmented matrix.

	System	Augmented matrix
	$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
Add $(-1)\times$ Equation 1 to Equation 2. Add $(-3)\times$ Equation 1 to Equation 3.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$
Multiply Equation 3 by $\frac{1}{2}$.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Add $(-3)\times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).	$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{R_2 - 3R_3 \to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Interchange Equations 2 and 3.	$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases}$	$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Now we use back-substitution to find that

$$z = 3 \qquad \begin{aligned} y - 2z &= 1 \\ y - 2(3) &= 1 \\ y - 6 &= 1 \\ y &= 7 \end{aligned} \qquad \begin{aligned} x - y + 3z &= 4 \\ x - 7 + 3(3) &= 4 \\ x + 2 &= 4 \\ x &= 2 \end{aligned}$$

The solution is $(2, 7, 3)$.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

Solution:

Linear System	*Associated Augmented Matrix*
$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$	$\left[\begin{array}{rrr\vdots r} 1 & -2 & 3 & 9 \\ -1 & 3 & 1 & -2 \\ 2 & -5 & 5 & 17 \end{array}\right]$
Add the first equation to the second equation.	Add the first row to the second row ($R_1 + R_2$).
$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases}$	$\begin{array}{r} \\ R_1 + R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrr\vdots r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 2 & -5 & 5 & 17 \end{array}\right]$
Add -2 times the first equation to the third equation.	Add -2 times the first row to the third row ($-2R_1 + R_3$).
$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases}$	$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \end{array}\left[\begin{array}{rrr\vdots r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & -1 & -1 & -1 \end{array}\right]$
Add the second equation to the third equation.	Add the second row to the third row ($R_2 + R_3$).
$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases}$	$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \end{array}\left[\begin{array}{rrr\vdots r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 3 & 6 \end{array}\right]$
Multiply the third equation by $\frac{1}{3}$.	Multiply the third row by $\frac{1}{3}\left(\frac{1}{3}R_3\right)$.
$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$	$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \end{array}\left[\begin{array}{rrr\vdots r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Now we use back-substitution to find that

$$z = 2 \qquad \begin{aligned} y + 4z &= 7 \\ y + 4(2) &= 7 \\ y + 8 &= 7 \\ y &= -1 \end{aligned} \qquad \begin{aligned} x - 2y + 3z &= 9 \\ x - 2(-1) + 3(2) &= 9 \\ x + 8 &= 9 \\ x &= 1 \end{aligned}$$

The solution is $(1, -1, 2)$.

EXAMPLE: Show that the following system has no solutions.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

Solution: We have

$$\begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{bmatrix} \xrightarrow{\frac{1}{18}R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we pick for x, y, and z, the last equation will never be a true statement. This means the system *has no solution.*

EXAMPLE: Show that the following system has infinitely many solutions.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

Solution: We have

$$\begin{bmatrix} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. Since the equation adds no new information about the variables, we can drop it from the system. So the last matrix corresponds to the system

$$\begin{cases} x \qquad - 7z = -5 & \text{Equation 1} \\ \qquad y - 3z = 1 & \text{Equation 2} \end{cases}$$

Leading variables

Now we solve for the leading variables x and y in terms of the non-leading variable z:

$$x = 7z - 5 \qquad \text{Solve for x in Equation 1}$$

$$y = 3z + 1 \qquad \text{Solve for y in Equation 2}$$

To obtain the complete solution, we let t represent any real number, and we express x, y, and z in terms of t:

$$\begin{cases} x = 7t - 5 \\ y = 3t + 1 \\ z = t \end{cases}$$

We can also write the solution as the ordered triple $(7t - 5, 3t + 1, t)$, where t is any real number.