Applications of Linear Algebra to Differential Equations

Consider the homogeneous linear system of differential equations

$$\frac{dx_1}{dt} = a_{11}x_1 + \ldots + a_{1n}x_n$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + \ldots + a_{nn}x_n$$

which can be rewritten as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

THEOREM 4 (Existence-Uniqueness Theorem): There exists one, and only one, solution of the initial-value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x^0} = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix} \tag{2}$$

Moreover, this solution exists for $-\infty < t < \infty$.

THEOREM 5: The dimension of the space V of all solutions of the homogeneous linear system of differential equations (1) is n.

THEOREM 6 (Test for Linear Independence): Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k$ be k solutions of

$$\dot{\mathbf{x}} = A\mathbf{x}$$

Select a convenient t_0. Then, $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k$ are linear independent solutions if and only if,

$$\mathbf{x}^1(t_0), \quad \mathbf{x}^2(t_0), \quad \ldots, \quad \mathbf{x}^k(t_0)$$

are linearly independent vectors in $\mathbb{R}^n$.

EXAMPLE: Consider the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -x_1 - 2x_2 \end{aligned} \tag{3}$$

or

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

This system of equations arose from the single second-order equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \tag{4}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{dy}{dt}$$

To find two linearly independent solutions of (4) we note that the characteristic polynomial is

$$P(r) = r^2 + 2r + 1 = (r+1)^2$$

Thus, $r = -1$ is a repeated root of the characteristic equation, and therefore

$$y_1(t) = e^{-t} \quad \text{and} \quad y_2(t) = te^{-t}$$

are two solutions of (4). We see that

$$\mathbf{x^1}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} e^{-t} \\ (e^{-t})' \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

and

$$\mathbf{x^2}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} te^{-t} \\ (te^{-t})' \end{pmatrix} = \begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix}$$

are two solutions of (3). To determine whether $\mathbf{x^1}$ and $\mathbf{x^2}$ are linearly dependent or linearly independent, we check whether their initial values

$$\mathbf{x^1}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x^2}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are linearly dependent or linearly independent vectors in $\mathbb{R}^2$. The easiest way to do that is to note that $\mathbf{x^1}(0)$ and $\mathbf{x^2}(0)$ are not multiples of each other and therefore linearly independent by the Corollary to Theorem 1 (Section 3.3). Here is another way: Consider the equation

$$\begin{aligned} c_1\mathbf{x^1}(0) + c_2\mathbf{x^2}(0) &= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ -c_1 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This equation implies that both c_1, and c_2 are zero. Hence, $\mathbf{x^1}(0)$ and $\mathbf{x^2}(0)$ are linearly independent vectors in $\mathbb{R}^2$. Consequently, by Theorem 6 above, $\mathbf{x^1}(t)$ and $\mathbf{x^2}(t)$ are linearly independent solutions of

(3). Therefore, by Theorem 5 above and Theorem 3 from Section 3.3, every solution $\mathbf{x}(t)$ of (3) can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x^1}(t) + c_2\mathbf{x^2}(t) = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix} = \begin{pmatrix} (c_1+c_2t)e^{-t} \\ (c_2-c_1-c_2t)e^{-t} \end{pmatrix} \tag{5}$$

REMARK: Another method of solving (3) will be discussed in Section 3.10.

EXAMPLE: Solve the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: From the Example above, every solution $\mathbf{x}(t)$ must be of the form (5). The constants c_1, and c_2 are determined from the initial conditions

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} c_1 \\ c_2 - c_1 \end{pmatrix}$$

Therefore, $c_1 = 1$ and $c_2 = 1 + c_1 = 2$. Hence

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} (1+2t)e^{-t} \\ (1-2t)e^{-t} \end{pmatrix}$$