Dimension of a Vector Space

DEFINITION: The vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors from a vector space V and $c_1, \ldots, c_p$ are scalars, is called a **linear combination** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

DEFINITION: The set of all combinations of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ from a vector space V is denoted by $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of V spanned** (or generated) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

EXAMPLE 1: The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^2$, since any vector

$$\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

from $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{e}_1, \mathbf{e}_2$:

$$\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 + 0 \cdot c_2 \\ 0 \cdot c_1 + 1 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 \\ 0 \cdot c_1 \end{bmatrix} + \begin{bmatrix} 0 \cdot c_2 \\ 1 \cdot c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$$

REMARK: In the same way one can show that the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^n$.

EXAMPLE 2: The vectors

$$1, \quad t, \quad t^2, \quad \ldots, \quad t^n$$

span $\mathbb{P}_n$ (the vector space of all polynomials of degree at most n).

EXAMPLE 3: Let V be the set of all solutions of the differential equation

$$\frac{d^2x}{dt^2} - x = 0$$

Let $x_1(t)$ be the function whose value at any time t is e^t and let $x_2(t)$ be the function whose value at any time t is e^{-t}. The functions $x_1(t)$ and $x_2(t)$ are in V since they satisfy the differential equation. Moreover, these functions also span V since every solution $x(t)$ of the differential equation can be written in the form

$$x(t) = c_1e^t + c_2e^{-t}$$

so that

$$x(t) = c_1x_1(t) + c_2x_2(t)$$

DEFINITION: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be linearly **dependent** if there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be linearly **independent** if the vector equation

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

THEOREM 1: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent if and only if at least one of these vectors is a linear combination of the others.

EXAMPLE 4: Show that the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent. Then show that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent.

Solution 1: To show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent, we find c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

that is,

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} c_1 - 2c_2 + 3c_3 \\ c_1 - c_2 + 5c_3 \\ 2c_1 - 4c_2 + 6c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This can be rewritten as the following system of equations:

$$\begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_1 - c_2 + 5c_3 = 0 \\ 2c_1 - 4c_2 + 6c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 + 7c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = -7c_3 \\ c_2 = -2c_3 \\ c_3 \text{ is free} \end{cases}$$

For example, if $c_3 = -1$, then $c_1 = 7$ and $c_2 = 2$, that is,

$$7\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0}$$

We now show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has only the trivial solution (which means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent). We have

$$\begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_1 - c_2 + 5c_3 = 0 \\ 2c_1 - 4c_2 + 6c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_2 + 2c_3 = 0 \\ c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

Solution 2 (Section 3.7 is required): We have

$$[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the number of pivots (two) is lesser than the number of columns (three), the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent. Similarly,

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the number of pivots (three) is equal to the number of columns (three), the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Solution 3 (Sections 3.5, 3.7 are required): We have

$$\det[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Since the determinant is equal to zero, the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent by Lemma 3 from Section 3.7. Similarly,

$$\det[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Since the determinant is not equal to zero, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent by Lemma 3 from Section 3.7.

EXAMPLE 5: The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent (see Appendix I).

DEFINITION: The **dimension** of a vector space V, denoted by $\dim V$, is the fewest number of linearly independent vectors which span V. V is said to be a **finite dimensional** space if its dimension is finite. On the other hand, V is said to be an **infinite dimensional** space if no set of finitely many elements span V.

THEOREM 2: If n linearly independent vectors span V, then $\dim V = n$.

EXAMPLE 6: The dimension of $\mathbb{R}^n$ is n, since the vectors

$$\mathbf{e}_1, \quad \mathbf{e}_2, \quad \ldots, \quad \mathbf{e}_n$$

are linearly independent and span $\mathbb{R}^n$.

EXAMPLE 7: The dimension of $\mathbb{P}^n$ is $n+1$, since the vectors

$$1, \quad t, \quad t^2, \quad \ldots, \quad t^n$$

span $\mathbb{P}_n$ and are linearly independent (because $c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n$ is identically equal to zero if and only if $c_0 = c_1 = c_2 = \ldots = c_n = 0$).

EXAMPLE 8: The dimension of the vector space of all 3×2 matrices is 6 (see Appendix II). In general, the dimension of the vector space of all $n \times m$ matrices is nm.

EXAMPLE 9: Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the same as in Example 4. Then

$$\dim\,(\text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = 3$$

$$\dim\,(\text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}) = 2$$

$$\dim\,(\text{Span}\,\{\mathbf{v}_1\}) = 1$$

since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. However, $\dim\,(\text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}) = 2$, since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent, $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent (because they are not multiples of each other) and $\text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$ (because $\mathbf{u}_3$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$).

DEFINITION: If a set of linearly independent vectors span a vector space V, then this set of vectors is said to be a **basis** for V. A basis may also be called a **coordinate system**.

EXAMPLE 10: The vectors

$$\mathbf{e}_1, \quad \mathbf{e}_2, \quad \ldots, \quad \mathbf{e}_n$$

are a basis (so-called **standard basis**) for $\mathbb{R}^n$. If

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$$

and relative to this basis the x_i are called “components” or “coordinates.”

COROLLARY: In a finite dimensional vector space, each basis has the same number of vectors, and this number is the dimension of the space.

THEOREM 3: Any n linearly independent vectors in an n dimensional space V must also span V. That is to say, any n linearly independent vectors in an n dimensional space V are a basis for V.

EXAMPLE 11: Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the same as in Example 4. By Theorem 3 above, these vectors form a basis for $\mathbb{R}^3$, since these *three* vectors are linearly independent and the dimension of $\mathbb{R}^3$ is also *three*.

Appendix I

The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are linearly independent, since

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \ldots + c_n\mathbf{e}_n = \mathbf{0}$$

has only the trivial solution. Indeed, the equation above can be rewritten as

$$c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + c_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1 \cdot c_1 + 0 \cdot c_2 + \ldots + 0 \cdot c_n \\ 0 \cdot c_1 + 1 \cdot c_2 + \ldots + 0 \cdot c_n \\ \vdots \\ 0 \cdot c_1 + 0 \cdot c_2 + \ldots + 1 \cdot c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

so, $c_1 = c_2 = \ldots = c_n = 0$.

Appendix II

To show that the dimension of the vector space V of all 3×2 matrices is 6, consider the following matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We first note that these matrices span V, because any 3×2 matrix can be written as a linear combination of $M_1, M_2, \ldots, M_6$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{31} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_{32} \end{bmatrix}$$

$$= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{31} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{32} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We also note that $M_1, M_2, \ldots, M_6$ are linearly independent. Indeed, suppose

$$c_1 M_1 + c_2 M_2 + \ldots + c_6 M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for some scalars $c_1, c_2, \ldots, c_6$. Since

$$\begin{aligned} & c_1 M_1 + c_2 M_2 + \ldots + c_6 M_6 \\ &= c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + c_6 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_5 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & c_6 \end{bmatrix} \\ &= \begin{bmatrix} c_1 & c_4 \\ c_2 & c_5 \\ c_3 & c_6 \end{bmatrix} \end{aligned}$$

it follows that

$$\begin{bmatrix} c_1 & c_4 \\ c_2 & c_5 \\ c_3 & c_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

therefore $c_1 = c_2 = \ldots = c_6 = 0$. This means that $M_1, M_2, \ldots, M_6$ are linearly independent. Since 6 linearly independent vectors span V, it follows that then $\dim V = 6$ by Theorem 2 above.