

Vector Spaces

DEFINITION: A **vector space** is a nonempty set V of objects, called **vectors**, on which are defined two operations, called **addition** and **multiplication by scalars** (real numbers), subject to the following 10 axioms (or rules):

- (A) The sum of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} + \mathbf{v}$, is in V .
- (B) The scalar multiple of \mathbf{u} by c , denoted by $c\mathbf{u}$, is in V .
- (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- (iii) There is a unique element $\mathbf{0}$ in V , called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- (iv) For each \mathbf{u} in V , there is a unique vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- (v) $1 \cdot \mathbf{u} = \mathbf{u}$.
- (vi) $a(b\mathbf{u}) = (ab)\mathbf{u}$.
- (vii) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- (viii) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

These axioms must hold for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and all scalars a and b .

EXAMPLE: Let \mathbb{R}^n be the set of all sequences

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Define $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ as the vector addition and scalar multiplication defined in Section 3.1, that is

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

Then \mathbb{R}^n is a vector space. In fact,

- (A) $\mathbf{u} + \mathbf{v}$ is in \mathbb{R}^n .
- (B) $c\mathbf{u}$ is in \mathbb{R}^n .

(i) Since $x_i + y_i = y_i + x_i$ for all real numbers x_i, y_i , it follows that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Indeed,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \\ \vdots \\ y_n + x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

In the same way one can prove (ii): Since $(x_i + y_i) + z_i = x_i + (y_i + z_i)$ for all real numbers x_i, y_i, z_i , it follows that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(iii) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$, since

$$\mathbf{u} + \mathbf{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{u}$$

(iv) For each $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in \mathbb{R}^n , there is the vector $-\mathbf{u} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ in \mathbb{R}^n such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, since

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ x_2 + (-x_2) \\ \vdots \\ x_n + (-x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

(v) $1 \cdot \mathbf{u} = \mathbf{u}$, since

$$1 \cdot \mathbf{u} = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot x_2 \\ \vdots \\ 1 \cdot x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{u}$$

(vi) $a(b\mathbf{u}) = (ab)\mathbf{u}$, since

$$\begin{aligned} a(b\mathbf{u}) &= a \left(b \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = a \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{bmatrix} \\ &= \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{bmatrix} = (ab) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (ab)\mathbf{u} \end{aligned}$$

(vii) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, since

$$\begin{aligned} a(\mathbf{u} + \mathbf{v}) &= a \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right) = a \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \\ &= \begin{bmatrix} a(x_1 + y_1) \\ a(x_2 + y_2) \\ \vdots \\ a(x_n + y_n) \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + ay_1 \\ ax_2 + ay_2 \\ \vdots \\ ax_n + ay_n \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \\ \vdots \\ ay_n \end{bmatrix} \\ &= a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + a \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = a\mathbf{u} + a\mathbf{v} \end{aligned}$$

(viii) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, since

$$\begin{aligned} (a + b)\mathbf{u} &= (a + b) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (a + b)x_1 \\ (a + b)x_2 \\ \vdots \\ (a + b)x_n \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + bx_2 \\ \vdots \\ ax_n + bx_n \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} \\ &= a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a\mathbf{u} + b\mathbf{u} \end{aligned}$$

EXAMPLE: Let V be the set of all $n \times m$ matrices

$$\begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \\ \cdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}$$

Define $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ in the following way:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \\ \cdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} + \begin{bmatrix} y_{11} & \cdots & y_{1m} \\ y_{21} & \cdots & y_{2m} \\ \cdots & & \\ y_{n1} & \cdots & y_{nm} \end{bmatrix} = \begin{bmatrix} x_{11} + y_{11} & \cdots & x_{1m} + y_{1m} \\ x_{21} + y_{21} & \cdots & x_{2m} + y_{2m} \\ \cdots & & \\ x_{n1} + y_{n1} & \cdots & x_{nm} + y_{nm} \end{bmatrix}$$

and

$$a\mathbf{u} = a \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \\ \cdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} = \begin{bmatrix} ax_{11} & \cdots & ax_{1m} \\ ax_{21} & \cdots & ax_{2m} \\ \cdots & & \\ ax_{n1} & \cdots & ax_{nm} \end{bmatrix}$$

Then V is a vector space. In fact,

(A) $\mathbf{u} + \mathbf{v}$ is in V .

(B) $c\mathbf{u}$ is in V .

The axioms (i) and (ii) are true, since

$$x_{ij} + y_{ij} = y_{ij} + x_{ij}$$

and

$$(x_{ij} + y_{ij}) + z_{ij} = x_{ij} + (y_{ij} + z_{ij})$$

for all real numbers x_{ij}, y_{ij}, z_{ij} .

(iii) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \cdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(iv) For each $\mathbf{u} = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \\ \cdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}$ in V , there is the vector $-\mathbf{u} = \begin{bmatrix} -x_{11} & \cdots & -x_{1m} \\ -x_{21} & \cdots & -x_{2m} \\ \cdots & & \\ -x_{n1} & \cdots & -x_{nm} \end{bmatrix}$ in V such that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

One can show that (v) – (viii) are also true.

EXAMPLE: The set \mathbb{P}_n of all polynomials of degree at most n :

$$\mathbf{p}(t) = a_n t^n + \cdots + a_2 t^2 + a_1 t + a_0$$

where the coefficients a_n, \dots, a_0 and the variable t are real numbers is a vector space.

EXAMPLE: The set of all real-valued functions defined on \mathbb{R} is a vector space.

EXAMPLE: Here are some examples of sets that are *not* vector spaces:

1. The set of all vectors

$$\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where } x_i \geq 0$$

is *not* a vector space, since axiom (B) fails.

2. The set of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} \quad \text{where } x, z \text{ are all real numbers}$$

is *not* a vector space (there is no $\mathbf{0}$). One can check, however, that the set of all vectors

$$\mathbf{v} = \langle x, 0, z \rangle$$

is a vector space (see Appendix I).

3. The set V of all vectors $\mathbf{u} = \langle x, 0, x^3 \rangle$ where x is any real number is *not* a vector space, since axiom (A) fails. Indeed, if $\mathbf{u}_1 = \langle 1, 0, 1 \rangle$ and $\mathbf{u}_2 = \langle 2, 0, 8 \rangle$, then $\mathbf{u}_1 + \mathbf{u}_2 = \langle 3, 0, 9 \rangle$ which is not in V .

4. The set of all polynomials of degree n with $n > 0$ is *not* a vector space (there is no $\mathbf{0}$).

DEFINITION: A **subspace** of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H .

2. H is closed under vector addition. That is, for each \mathbf{u} and \mathbf{v} in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .

3. H is closed under multiplication by scalars. That is, for each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

THEOREM 1: A subspace H of a vector space V is a vector space.

EXAMPLE: Let H be the set of all functions $x(t)$ which satisfy the differential equation

$$\frac{d^2x}{dt^2} - x = 0 \tag{1}$$

with the sum of two functions and the product of a function by a number being defined in the usual manner. That is to say,

$$(f_1 + f_2)(t) = f_1(t) + f_2(t) \quad \text{and} \quad (cf)(t) = cf(t)$$

It is easy to verify that H is a vector space. We first note that H is a subset of the vector space of all real-valued functions. Let us show that this is a subspace. Indeed,

1. $\mathbf{0}$ is from H , since $x(t) = 0$ is a solution of (1).

2. If $x_1(t)$ and $x_2(t)$ are in H , then $x_1(t) + x_2(t)$ is in H , since the differential equation (1) is linear.

3. Similarly, if $x(t)$ is in H , then $cx(t)$ is in H .

So, H is a subspace of the vector space of all real-valued functions. Therefore H is a vector space by Theorem 1 above.

EXAMPLE: Let H be the set of all functions $x(t)$ which satisfy the differential equation

$$\frac{d^2x}{dt^2} - 6x^2 = 0$$

with the sum of two functions and the product of a function by a number being defined in the usual manner. H is not a vector space since the sum of any two solutions, while being defined, is not necessarily in H . Similarly, the product of a solution by a constant is not necessarily in H . For example, the function

$$x(t) = \frac{1}{t^2}$$

is in H since it satisfies the differential equation (see Appendix II), but the function

$$2x(t) = \frac{2}{t^2}$$

is not in H since it does not satisfy the differential equation.

THEOREM 2: If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V , then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$

$$c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

is a subspace of V (and therefore a vector space).

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The set of all linear combinations of \mathbf{v}_1 and \mathbf{v}_2

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

is a vector space by Theorem 2 above.

EXAMPLE: Let H be the set of all vectors of the form

$$\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix}$$

where a, b , and c are arbitrary scalars. Show that H is a vector space.

Solution: We have

$$\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix} = \begin{bmatrix} 1 \cdot a + (-1) \cdot b + 0 \cdot c \\ 0 \cdot a + 1 \cdot b + (-1) \cdot c \\ (-1) \cdot a + 0 \cdot b + 1 \cdot c \\ 0 \cdot a + 1 \cdot b + 0 \cdot c \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} + c \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3}$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in the vector space \mathbb{R}^4 and H is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, it follows that H is a vector space by Theorem 2 above.

Appendix I

Here we show that the set H of all vectors $\langle x, 0, z \rangle$, where x, z are real numbers, is a vector space. Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} x_3 \\ 0 \\ z_3 \end{bmatrix}$$

Solution 1:

(A) $\mathbf{u} + \mathbf{v}$ is in H , since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix}$$

(B) $c\mathbf{u}$ is in H , since

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \\ cz_1 \end{bmatrix}$$

(i) We have

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ 0 \\ z_2 + z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

In the same way one can prove (ii): Since $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, it follows that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(iii) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$, since

$$\mathbf{u} + \mathbf{0} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ 0 + 0 \\ z_1 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{u}$$

(iv) For each $\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}$ in H , there is the vector $-\mathbf{u} = \begin{bmatrix} -x_1 \\ 0 \\ -z_1 \end{bmatrix}$ in H such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, since

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ 0 \\ -z_1 \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ 0 + 0 \\ z_1 + (-z_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

(v) $1 \cdot \mathbf{u} = \mathbf{u}$, since

$$1 \cdot \mathbf{u} = 1 \cdot \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot 0 \\ 1 \cdot z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{u}$$

(vi) $a(b\mathbf{u}) = (ab)\mathbf{u}$, since

$$\begin{aligned} a(b\mathbf{u}) &= a\left(b\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}\right) = a\begin{bmatrix} bx_1 \\ 0 \\ bz_1 \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ 0 \\ a(bz_1) \end{bmatrix} \\ &= \begin{bmatrix} (ab)x_1 \\ 0 \\ (ab)z_1 \end{bmatrix} = (ab)\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = (ab)\mathbf{u} \end{aligned}$$

(vii) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, since

$$\begin{aligned} a(\mathbf{u} + \mathbf{v}) &= a\left(\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}\right) = a\begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix} \\ &= \begin{bmatrix} a(x_1 + x_2) \\ 0 \\ a(z_1 + z_2) \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + ax_2 \\ 0 \\ az_1 + az_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ 0 \\ az_1 \end{bmatrix} + \begin{bmatrix} ax_2 \\ 0 \\ az_2 \end{bmatrix} \\ &= a\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = a\mathbf{u} + a\mathbf{v} \end{aligned}$$

(viii) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, since

$$\begin{aligned} (a + b)\mathbf{u} &= (a + b)\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} (a + b)x_1 \\ 0 \\ (a + b)z_1 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + bx_1 \\ 0 \\ az_1 + bz_1 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ 0 \\ az_1 \end{bmatrix} + \begin{bmatrix} bx_1 \\ 0 \\ bz_1 \end{bmatrix} \\ &= a\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + b\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = a\mathbf{u} + b\mathbf{u} \end{aligned}$$

Solution 2: We first show that the set H of all vectors $\langle x, 0, z \rangle$, where x, z are real numbers, is a subspace of \mathbb{R}^3 . Indeed,

1. The zero vector $\langle 0, 0, 0 \rangle$ of \mathbb{R}^3 is obviously in H .

2. If \mathbf{u} and \mathbf{v} are in H , then $\mathbf{u} + \mathbf{v}$ is in H , since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix}$$

3. Similarly, if \mathbf{u} is in H , then $c\mathbf{u}$ is in H , since

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \\ cz_1 \end{bmatrix}$$

So, H is a subspace of the vector space \mathbb{R}^3 by the definition of a subspace. Therefore H is a vector space by Theorem 1 above.

Solution 3: We have

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot z \\ 0 \cdot x + 0 \cdot z \\ 0 \cdot x + 1 \cdot z \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + z \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space \mathbb{R}^3 and H is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that H is a vector space by Theorem 2.

Appendix II

One can easily check that the function $x(t) = \frac{1}{t^2}$ satisfies

$$\frac{d^2x}{dt^2} - 6x^2 = 0$$

plugging it into the equation. Here we show how to come up with this function without guessing. To this end, we first rewrite it as

$$\frac{d^2x}{dt^2} = 6x^2$$

Multiplying both sides by $\frac{dx}{dt}$, we get

$$\frac{d^2x}{dt^2} \cdot \frac{dx}{dt} = 6x^2 \cdot \frac{dx}{dt}$$

By the Chain Rule, the left-hand side is $\frac{d}{dt} \left(\frac{1}{2} \left(\frac{dx}{dt} \right)^2 \right)$ and the right-hand side is $\frac{d}{dt} (2x^3)$. So, we have

$$\frac{d}{dt} \left(\frac{1}{2} \left(\frac{dx}{dt} \right)^2 \right) = \frac{d}{dt} (2x^3)$$

Integrating both sides, we obtain

$$\frac{1}{2} \left(\frac{dx}{dt} \right)^2 = 2x^3$$

$$\left(\frac{dx}{dt} \right)^2 = 4x^3$$

$$\frac{dx}{dt} = 2x^{3/2}$$

$$\frac{dx}{x^{3/2}} = 2dt$$

$$\int \frac{dx}{x^{3/2}} = \int 2dt$$

$$\frac{x^{-3/2+1}}{-3/2+1} = 2t$$

$$\frac{x^{-1/2}}{-1/2} = 2t$$

$$x^{-1/2} = -t$$

$$(x^{-1/2})^2 = (-t)^2$$

$$x^{-1} = t^2$$

$$x = t^{-2} = \frac{1}{t^2}$$

Appendix III

EXAMPLE: Let H be the set of all vectors of the form

$$\begin{bmatrix} 4a - b \\ 2b \\ a - 2b \\ a - b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is a vector space.

Solution: We have

$$\begin{bmatrix} 4a - b \\ 2b \\ a - 2b \\ a - b \end{bmatrix} = a \underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space \mathbb{R}^4 and H is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that H is a vector space by Theorem 2.

EXAMPLE: Let H be the set of all vectors of the form

$$\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is not a vector space.

Solution: H is not a vector space, since $\mathbf{0} \notin H$ (the second entry is always nonzero).

EXAMPLE: Let V be the set of vectors

$$\begin{bmatrix} 2a \\ ab \\ 3b \end{bmatrix}$$

in \mathbb{R}^3 where a and b are real numbers. Then

- (A) V is a vector space, since $\mathbf{0}$ is in V
- (B) V is not a vector space, since $\mathbf{0}$ is not in V
- (C) V is a vector space, since $\mathbf{u} + \mathbf{v}$ is from V for any \mathbf{u} and \mathbf{v} from V
- (D) V is a vector space, since $c\mathbf{u}$ is from V for any \mathbf{u} from V and any c from \mathbb{R}
- (E) V is not a vector space, since there exist \mathbf{u} and \mathbf{v} from V such that $\mathbf{u} + \mathbf{v}$ is not from V
- (F) None of the above

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad (\text{here } a = 1, b = 1)$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \quad (\text{here } a = 2, b = 2)$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2+4 \\ 1+4 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix}$$

However,

$$\begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 2a \\ ab \\ 3b \end{bmatrix}$$

since $6 = 2a$ implies $a = 3$, $9 = 3b$ implies $b = 3$, therefore $5 \neq ab$. So, the correct answer is E.

Info: This problem was given in Spring 2017 (Midterm II). The average in the class for this problem was 60.2%.

EXAMPLE: Let V be the set of vectors

$$\begin{bmatrix} ab \\ a \\ b \end{bmatrix}$$

in \mathbb{R}^3 where a and b are real numbers. Show that V is not a vector space.

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (\text{here } a = 1, b = 1)$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \quad (\text{here } a = 2, b = 2)$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 1+2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

However,

$$\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} ab \\ a \\ b \end{bmatrix}$$

since otherwise $a = 3$, $b = 3$ and $ab = 5$, which is impossible.

EXAMPLE: Let V be the set of vectors

$$\begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$$

in \mathbb{R}^3 where a and b are real numbers. Show that V is not a vector space.

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (\text{here } a = 1, b = 1)$$

and

$$\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \quad (\text{here } a = 2, b = -1)$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + (-2) \\ 0 + 0 \\ 1 + (-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

However,

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$$

since otherwise $b = 0$ and $ab = -1$, which is impossible.

REMARK 1: Note that if we set $a = 1, b = 1$ and $a = 2, b = 2$ for \mathbf{u} and \mathbf{v} (like we did in the two previous examples), we will *not* get a contradiction. Indeed, in this case we have

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

But

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + 4 \\ 0 + 0 \\ 1 + 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

which is equal to $\begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$ if $a = 5/3$ and $b = 3$.

REMARK 2: If V is the set of vectors

$$\begin{bmatrix} a+b \\ 0 \\ b \end{bmatrix}$$

in \mathbb{R}^3 where a and b are real numbers, then V is a vector space. Indeed, we have

$$\begin{bmatrix} a+b \\ 0 \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space \mathbb{R}^3 and V is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that V is a vector space by Theorem 2.

EXAMPLE: Let W_1 be the set of vectors

$$\mathbf{u} = \begin{bmatrix} x \\ x+1 \\ y-1 \end{bmatrix}$$

and let W_2 be the set of vectors

$$\mathbf{v} = \begin{bmatrix} x \\ y^2 \\ x+y \end{bmatrix}$$

where x and y are all real numbers. Then

- (A) W_1 is a vector space, but W_2 is not a vector space.
- (B) W_1 is not a vector space, but W_2 is a vector space.
- (C) W_1 is a vector space and W_2 is a vector space.
- (D) W_1 is not a vector space and W_2 is not a vector space. ← correct

Info: This problem was given in Fall 2017 (Midterm Exam II). The average in the class for this problem was 71.4%.

EXAMPLE: Let W be the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt + 1$, where a, b are real numbers. Then

- (A) W is not a vector space, since it is not closed under multiplication by a scalar.
- (B) W is not a vector space, since $\mathbf{0}$ is not in W .
- (C) W is not a vector space, since it is not closed under addition.
- (D) All of the above ← correct
- (E) None of the above

Info: This problem was given in Fall 2017 (Final Exam II). The average in the class for this problem was 43.2%.

REMARK: If W is the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt$, where a, b are real numbers, then W is a vector space. Indeed, putting

$$\mathbf{v}_1 = t^2 \quad \text{and} \quad \mathbf{v}_2 = t$$

we get

$$\mathbf{p}(t) = a\mathbf{v}_1 + b\mathbf{v}_2$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space \mathbb{P}_2 and W is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that W is a vector space by Theorem 2.

EXAMPLE: Let W be the set of singular 2×2 matrices under the usual operations. Then

- Ⓐ W is not a vector space, since it is not closed under addition. ← correct
- Ⓑ W is not a vector space, since $\mathbf{0}$ is not in W .
- Ⓒ W is not a vector space, since it is not closed under multiplication by a scalar.
- Ⓓ W is not a vector space, since it is not a subspace of the vector space of 2×2 matrices.
- Ⓔ None of the above

Info: This problem was given in Fall 2017 (Final Exam I). The average in the class for this problem was 34.3%.

REMARK: A singular matrix is a square matrix that does not have a matrix inverse. A matrix is singular if and only if its determinant is 0 (see Sections 3.5, 3.6).