# Vector Spaces

DEFINITION: A **vector space** is a nonempty set $V$ of objects, called **vectors**, on which are defined two operations, called **addition** and **multiplication by scalars** (real numbers), subject to the following 10 axioms (or rules):

(A) The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in $V$.

(B) The scalar multiple of $\mathbf{u}$ by $c$, denoted by $c\mathbf{u}$, is in $V$.

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

(iii) There is a unique element $\mathbf{0}$ in $V$, called the zero vector, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

(iv) For each $\mathbf{u}$ in $V$, there is a unique vector $-\mathbf{u}$ in $V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(v) $1\cdot\mathbf{u}=\mathbf{u}$.

(vi) $a(b\mathbf{u})=(ab)\mathbf{u}$.

(vii) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$.

(viii) $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$.

These axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and all scalars $a$ and $b$.

EXAMPLE: Let $\mathbb{R}^n$ be the set of all $n\times 1$ matrices

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Define $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ as the vector addition and scalar multiplication defined in Section 3.1, that is

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}=\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

and

$$c\mathbf{u}=c\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

Then $\mathbb{R}^n$ is a vector space. In fact,

(A) $\mathbf{u}+\mathbf{v}$ is in $\mathbb{R}^n$.

(B) $c\mathbf{u}$ is in $\mathbb{R}^n$.

(i) Since $x_i + y_i = y_i + x_i$ for all real numbers $x_i, y_i$, it follows that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Indeed,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \\ \vdots \\ y_n + x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

In the same way one can prove (ii): Since $(x_i + y_i) + z_i = x_i + (y_i + z_i)$ for all real numbers $x_i, y_i, z_i$, it follows that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(iii) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$, since

$$\mathbf{u} + \mathbf{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{u}$$

(iv) For each $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$, there is the vector $-\mathbf{u} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ in $\mathbb{R}^n$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, since

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ x_2 + (-x_2) \\ \vdots \\ x_n + (-x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

(v) $1 \cdot \mathbf{u} = \mathbf{u}$, since

$$1 \cdot \mathbf{u} = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot x_2 \\ \vdots \\ 1 \cdot x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{u}$$

(vi) $a(b\mathbf{u}) = (ab)\mathbf{u}$, since

$$a(b\mathbf{u}) = a \left( b \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = a \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{bmatrix}$$

$$= \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{bmatrix} = (ab) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (ab)\mathbf{u}$$

(vii) $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$, since

$$a(\mathbf{u}+\mathbf{v}) = a\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}\right) = a\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

$$= \begin{bmatrix} a(x_1+y_1) \\ a(x_2+y_2) \\ \vdots \\ a(x_n+y_n) \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2 \\ \vdots \\ ax_n+ay_n \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \\ \vdots \\ ay_n \end{bmatrix}$$

$$= a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + a\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = a\mathbf{u}+a\mathbf{v}$$

(viii) $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$, since

$$(a+b)\mathbf{u} = (a+b)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \\ \vdots \\ (a+b)x_n \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+bx_1 \\ ax_2+bx_2 \\ \vdots \\ ax_n+bx_n \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix}$$

$$= a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a\mathbf{u}+b\mathbf{u}$$

EXAMPLE: Let $V$ be the set of all $n \times m$ matrices

$$\begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$$

Define $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ in the following way:

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix} + \begin{bmatrix} y_{11} & \dots & y_{1m} \\ y_{21} & \dots & y_{2m} \\ \dots & & \\ y_{n1} & \dots & y_{nm} \end{bmatrix} = \begin{bmatrix} x_{11}+y_{11} & \dots & x_{1m}+y_{1m} \\ x_{21}+y_{21} & \dots & x_{2m}+y_{2m} \\ \dots & & \\ x_{n1}+y_{n1} & \dots & x_{nm}+y_{nm} \end{bmatrix}$$

and

$$a\mathbf{u} = a\begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix} = \begin{bmatrix} ax_{11} & \dots & ax_{1m} \\ ax_{21} & \dots & ax_{2m} \\ \dots & & \\ ax_{n1} & \dots & ax_{nm} \end{bmatrix}$$

Then $V$ is a vector space. In fact,
(A) $\mathbf{u}+\mathbf{v}$ is in $V$.

(B) $c\mathbf{u}$ is in $V$.

The axioms (i) and (ii) are true, since

$$x_{ij}+y_{ij} = y_{ij}+x_{ij}$$

and

$$(x_{ij}+y_{ij})+z_{ij} = x_{ij}+(y_{ij}+z_{ij})$$

for all real numbers $x_{ij}, y_{ij}, z_{ij}$.

(iii) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \dots & & \\ 0 & \dots & 0 \end{bmatrix}$$

satisfies $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

(iv) For each $\mathbf{u} = \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$ in $V$, there is the vector $-\mathbf{u} = \begin{bmatrix} -x_{11} & \dots & -x_{1m} \\ -x_{21} & \dots & -x_{2m} \\ \dots & & \\ -x_{n1} & \dots & -x_{nm} \end{bmatrix}$ in $V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

One can show that (v) – (viii) are also true.

EXAMPLE: The set $\mathbb{P}_n$ of all polynomials of degree at most $n$:

$$\mathbf{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable $t$ are real numbers is a vector space.

EXAMPLE: The set of all real-valued functions defined on $\mathbb{R}$ is a vector space.

EXAMPLE: Here are some examples of sets that are *not* vector spaces:

1. The set of all vectors

$$\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_i \geq 0$$

is *not* a vector space, since axiom (B) fails.

2. The set of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} \quad \text{where } x, z \text{ are all real numbers}$$

is *not* a vector space (there is no $\mathbf{0}$). One can check, however, that the set of all vectors

$$\mathbf{v} =< x, 0, z >$$

*is* a vector space (see Appendix I).

3. The set $V$ of all vectors $\mathbf{u} =< x, 0, x^3 >$ where $x$ is any real number is *not* a vector space, since axiom (A) fails. Indeed, if $\mathbf{u_1} =< 1, 0, 1 >$ and $\mathbf{u_2} =< 2, 0, 8 >$, then $\mathbf{u_1} + \mathbf{u_2} =< 3, 0, 9 >$ which is not in $V$.

4. The set of all polynomials of degree $n$ with $n > 0$ is *not* a vector space (there is no $\mathbf{0}$).

DEFINITION: A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has 3 properties:

1. The zero vector of $V$ is in $H$.

2. $H$ is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in $H$, the sum $\mathbf{u} + \mathbf{v}$ is in $H$.

3. $H$ is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in $H$ and each scalar $c$, the vector $c\mathbf{u}$ is in $H$.

THEOREM 1: A subspace $H$ of a vector space $V$ is a vector space.

EXAMPLE: Let $H$ be the set of all functions $x(t)$ which satisfy the differential equation

$$\frac{d^2x}{dt^2} - x = 0 \tag{1}$$

with the sum of two functions and the product of a function by a number being defined in the usual manner. That is to say,

$$(f_1 + f_2)(t) = f_1(t) + f_2(t) \qquad \text{and} \qquad (cf)(t) = cf(t)$$

It is easy to verify that $H$ is a vector space. We first note that $H$ is a subset of the vector space of all real-valued functions. Let us show that this is a subspace. Indeed,

1. $\mathbf{0}$ is from $H$, since $x(t) = 0$ is a solution of (1).

2. If $x_1(t)$ and $x_2(t)$ are in $H$, then $x_1(t) + x_2(t)$ is in $H$, since the differential equation (1) is linear.

3. Similarly, if $x(t)$ is in $H$, then $cx(t)$ is in $H$.

So, $H$ is a subspace of the vector space of all real-valued functions. Therefore $H$ is a vector space by Theorem 1 above.

EXAMPLE: Let $H$ be the set of all functions $x(t)$ which satisfy the differential equation

$$\frac{d^2x}{dt^2} - 6x^2 = 0$$

with the sum of two functions and the product of a function by a number being defined in the usual manner. $H$ is not a vector space since the sum of any two solutions, while being defined, is not necessarily in $H$. Similarly, the product of a solution by a constant is not necessarily in $H$. For example, the function

$$x(t) = \frac{1}{t^2}$$

is in $H$ since it satisfies the differential equation (see Appendix II), but the function

$$2x(t) = \frac{2}{t^2}$$

is not in $H$ since it does not satisfy the differential equation.

THEOREM 2: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space $V$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

is a subspace of $V$ (and therefore a vector space).

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The set of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

is a vector space by Theorem 2 above.

EXAMPLE: Let $H$ be the set of all vectors of the form

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$$

where $a, b$, and $c$ are arbitrary scalars. Show that $H$ is a vector space.

Solution: We have

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = \begin{bmatrix} 1\cdot a & + & (-1)\cdot b & + & 0\cdot c \\ 0\cdot a & + & 1\cdot b & + & (-1)\cdot c \\ (-1)\cdot a & + & 0\cdot b & + & 1\cdot c \\ 0\cdot a & + & 1\cdot b & + & 0\cdot c \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} + c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3}$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in the vector space $\mathbb{R}^4$ and $H$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, it follows that $H$ is a vector space by Theorem 2 above.

# Appendix I

Here we show that the set $H$ of all vectors $< x, 0, z >$, where $x, z$ are real numbers, is a vector space. Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} x_3 \\ 0 \\ z_3 \end{bmatrix}$$

Solution 1:

(A) $\mathbf{u} + \mathbf{v}$ is in $H$, since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix}$$

(B) $c\mathbf{u}$ is in $H$, since

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \\ cz_1 \end{bmatrix}$$

(i) We have

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ 0 \\ z_2 + z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

In the same way one can prove (ii): Since $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, it follows that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(iii) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$, since

$$\mathbf{u} + \mathbf{0} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ 0 + 0 \\ z_1 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{u}$$

(iv) For each $\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}$ in $H$, there is the vector $-\mathbf{u} = \begin{bmatrix} -x_1 \\ 0 \\ -z_1 \end{bmatrix}$ in $H$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, since

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ 0 \\ -z_1 \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ 0 + 0 \\ z_1 + (-z_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

(v) $1 \cdot \mathbf{u} = \mathbf{u}$, since

$$1 \cdot \mathbf{u} = 1 \cdot \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot 0 \\ 1 \cdot z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{u}$$

(vi) $a(b\mathbf{u}) = (ab)\mathbf{u}$, since

$$a(b\mathbf{u}) = a\left(b\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}\right) = a\begin{bmatrix} bx_1 \\ 0 \\ bz_1 \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ 0 \\ a(bz_1) \end{bmatrix}$$
$$= \begin{bmatrix} (ab)x_1 \\ 0 \\ (ab)z_1 \end{bmatrix} = (ab)\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = (ab)\mathbf{u}$$

(vii) $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, since

$$a(\mathbf{u}+\mathbf{v}) = a\left(\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}\right) = a\begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix}$$
$$= \begin{bmatrix} a(x_1 + x_2) \\ 0 \\ a(z_1 + z_2) \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 + ax_2 \\ 0 \\ az_1 + az_2 \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 \\ 0 \\ az_1 \end{bmatrix} + \begin{bmatrix} ax_2 \\ 0 \\ az_2 \end{bmatrix}$$
$$= a\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = a\mathbf{u} + a\mathbf{v}$$

(viii) $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, since

$$(a+b)\mathbf{u} = (a+b)\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} (a+b)x_1 \\ 0 \\ (a+b)z_1 \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 + bx_1 \\ 0 \\ az_1 + bz_1 \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 \\ 0 \\ az_1 \end{bmatrix} + \begin{bmatrix} bx_1 \\ 0 \\ bz_1 \end{bmatrix}$$
$$= a\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + b\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = a\mathbf{u} + b\mathbf{u}$$

Solution 2: We first show that the set $H$ of all vectors $< x, 0, z >$, where $x, z$ are real numbers, is a subspace of $\mathbb{R}^3$. Indeed,

1. The zero vector $< 0, 0, 0 >$ of $\mathbb{R}^3$ is obviously in $H$.

2. If $\mathbf{u}$ and $\mathbf{v}$ are in $H$, then $\mathbf{u} + \mathbf{v}$ is in $H$, since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix}$$

3. Similarly, if $\mathbf{u}$ is in $H$, then $c\mathbf{u}$ is in $H$, since

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \\ cz_1 \end{bmatrix}$$

So, $H$ is a subspace of the vector space $\mathbb{R}^3$ by the definition of a subspace. Therefore $H$ is a vector space by Theorem 1 above.

Solution 3: We have

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 1 \cdot x & + & 0 \cdot z \\ 0 \cdot x & + & 0 \cdot z \\ 0 \cdot x & + & 1 \cdot z \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + z \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{R}^3$ and $H$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that $H$ is a vector space by Theorem 2.

# Appendix II

One can easily check that the function $x(t) = \dfrac{1}{t^2}$ satisfies

$$\frac{d^2x}{dt^2} - 6x^2 = 0$$

plugging it into the equation. Here we show how to come up with this function without guessing. To this end, we first rewrite it as

$$\frac{d^2x}{dt^2} = 6x^2$$

Multiplying both sides by $\dfrac{dx}{dt}$, we get

$$\frac{d^2x}{dt^2} \cdot \frac{dx}{dt} = 6x^2 \cdot \frac{dx}{dt}$$

By the Chain Rule, the left-hand side is $\dfrac{d}{dt}\left(\dfrac{1}{2}\left(\dfrac{dx}{dt}\right)^2\right)$ and the right-hand side is $\dfrac{d}{dt}\left(2x^3\right)$. So, we have

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right) = \frac{d}{dt}\left(2x^3\right)$$

Integrating both sides, we obtain

$$\begin{aligned}
\frac{1}{2}\left(\frac{dx}{dt}\right)^2 &= 2x^3 \\
\left(\frac{dx}{dt}\right)^2 &= 4x^3 \\
\frac{dx}{dt} &= 2x^{3/2} \\
\frac{dx}{x^{3/2}} &= 2dt \\
\int \frac{dx}{x^{3/2}} &= \int 2dt \\
\frac{x^{-3/2+1}}{-3/2+1} &= 2t \\
\frac{x^{-1/2}}{-1/2} &= 2t \\
x^{-1/2} &= -t \\
(x^{-1/2})^2 &= (-t)^2 \\
x^{-1} &= t^2 \\
x = t^{-2} &= \frac{1}{t^2}
\end{aligned}$$

# Appendix III

EXAMPLE: Let $H$ be the set of all vectors of the form

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix}$$

where $a$ and $b$ are arbitrary scalars. Show that $H$ is a vector space.

Solution: We have

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix} = a\underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + b\underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{R}^4$ and $H$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that $H$ is a vector space by Theorem 2.

EXAMPLE: Let $H$ be the set of all vectors of the form

$$\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$$

where $a$ and $b$ are arbitrary scalars. Show that $H$ is not a vector space.

Solution: $H$ is not a vector space, since $\mathbf{0} \notin H$ (the second entry is always nonzero).

EXAMPLE: Let $V$ be the set of vectors

$$\begin{bmatrix} 2a \\ ab \\ 3b \end{bmatrix}$$

in $\mathbb{R}^3$ where $a$ and $b$ are real numbers. Then

- (A) $V$ is a vector space, since $\mathbf{0}$ is in $V$
- (B) $V$ is not a vector space, since $\mathbf{0}$ is not in $V$
- (C) $V$ is a vector space, since $\mathbf{u}+\mathbf{v}$ is from $V$ for any $\mathbf{u}$ and $\mathbf{v}$ from $V$
- (D) $V$ is a vector space, since $c\mathbf{u}$ is from $V$ for any $\mathbf{u}$ from $V$ and any $c$ from $\mathbb{R}$
- (E) $V$ is not a vector space, since there exist $\mathbf{u}$ and $\mathbf{v}$ from $V$ such that $\mathbf{u}+\mathbf{v}$ is not from $V$
- (F) None of the above

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{(here } a=1, b=1\text{)}$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \quad \text{(here } a = 2, b = 2\text{)}$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2+4 \\ 1+4 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix}$$

However,

$$\begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 2a \\ ab \\ 3b \end{bmatrix}$$

since $6 = 2a$ implies $a = 3$, $9 = 3b$ implies $b = 3$, therefore $5 \neq ab$. So, the correct answer is E.

**Info**: This problem was given in Spring 2017 (Midterm II). The average in the class for this problem was 60.2%.

EXAMPLE: Let $V$ be the set of vectors

$$\begin{bmatrix} ab \\ a \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where $a$ and $b$ are real numbers. Show that $V$ is not a vector space.

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(here } a = 1, b = 1\text{)}$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \quad \text{(here } a = 2, b = 2\text{)}$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 1+2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

However,

$$\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} ab \\ a \\ b \end{bmatrix}$$

since otherwise $a = 3$, $b = 3$ and $ab = 5$, which is impossible.

EXAMPLE: Let $V$ be the set of vectors

$$\begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where $a$ and $b$ are real numbers. Show that $V$ is not a vector space.

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad (\text{here } a = 1, b = 1)$$

and

$$\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \qquad (\text{here } a = 2, b = -1)$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + (-2) \\ 0 + 0 \\ 1 + (-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

However,

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$$

since otherwise $b = 0$ and $ab = -1$, which is impossible.

REMARK 1: Note that if we set $a = 1, b = 1$ and $a = 2, b = 2$ for $\mathbf{u}$ and $\mathbf{v}$ (like we did in the two previous examples), we will *not* get a contradiction. Indeed, in this case we have

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

But

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + 4 \\ 0 + 0 \\ 1 + 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

which is equal to $\begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$ if $a = 5/3$ and $b = 3$.

REMARK 2: If $V$ is the set of vectors

$$\begin{bmatrix} a+b \\ 0 \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where $a$ and $b$ are real numbers, then $V$ *is* a vector space. Indeed, we have

$$\begin{bmatrix} a+b \\ 0 \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{R}^3$ and $V$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that $V$ is a vector space by Theorem 2.

EXAMPLE: Let $W_1$ be the set of vectors

$$\mathbf{u} = \begin{bmatrix} x \\ x+1 \\ y-1 \end{bmatrix}$$

and let $W_2$ be the set of vectors

$$\mathbf{v} = \begin{bmatrix} x \\ y^2 \\ x+y \end{bmatrix}$$

where $x$ and $y$ are all real numbers. Then

Ⓐ $W_1$ is a vector space, but $W_2$ is not a vector space.

Ⓑ $W_1$ is not a vector space, but $W_2$ is a vector space.

Ⓒ $W_1$ is a vector space and $W_2$ is a vector space.

Ⓓ $W_1$ is a not vector space and $W_2$ is not a vector space. $\longleftarrow$ correct

**Info**: This problem was given in Fall 2017 (Midterm Exam II). The average in the class for this problem was 71.4%.

EXAMPLE: Let $W$ be the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt + 1$, where $a, b$ are real numbers. Then

Ⓐ $W$ is not a vector space, since it is not closed under multiplication by a scalar.

Ⓑ $W$ is not a vector space, since $\mathbf{0}$ is not in $W$.

Ⓒ $W$ is not a vector space, since it is not closed under addition.

Ⓓ All of the above $\longleftarrow$ correct

Ⓔ None of the above

**Info**: This problem was given in Fall 2017 (Final Exam II). The average in the class for this problem was 43.2%.

REMARK: If $W$ is the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt$, where $a, b$ are real numbers, then $W$ *is* a vector space. Indeed, putting

$$\mathbf{v}_1 = t^2 \quad \text{and} \quad \mathbf{v}_2 = t$$

we get

$$\mathbf{p}(t) = a\mathbf{v}_1 + b\mathbf{v}_2$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{P}_2$ and $W$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that $W$ is a vector space by Theorem 2.

EXAMPLE: Let $W$ be the set of singular $2 \times 2$ matrices under the usual operations. Then

(A) $W$ is not a vector space, since it is not closed under addition. $\longleftarrow$ correct

(B) $W$ is not a vector space, since $\mathbf{0}$ is not in $W$.

(C) $W$ is not a vector space, since it is not closed under multiplication by a scalar.

(D) $W$ is not a vector space, since it is not a subspace of the vector space of $2 \times 2$ matrices.

(E) None of the above

**Info**: This problem was given in Fall 2017 (Final Exam I). The average in the class for this problem was 34.3%.

REMARK: A singular matrix is a square matrix that does not have a matrix inverse. A matrix is singular if and only if its determinant is 0 (see Sections 3.5, 3.6).