Fundamental Matrix Solutions; e^{At}

If $\mathbf{x^1}(t), \ldots, \mathbf{x^n}(t)$ are n linearly independent solutions of the differential equation

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

then every solution $\mathbf{x}(t)$ can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x^1}(t) + c_2\mathbf{x^2}(t) + \ldots + c_n\mathbf{x^n}(t) \tag{2}$$

Let $\mathbf{X}(t)$ be the matrix whose columns are $\mathbf{x^1}(t), \ldots, \mathbf{x^n}(t)$. Then, equation (2) can be written in the concise form $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}$, where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

DEFINITION: A matrix $\mathbf{X}(t)$ is called a **fundamental matrix solution** of (1) if its columns form a set of n linearly independent solutions of (1).

EXAMPLE: Find a fundamental matrix solution of the system of differential equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x} \tag{3}$$

Solution: We showed in Section 3.8 that

$$e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \quad e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

are three linearly independent solutions of (3). Hence

$$\mathbf{X}(t) = \begin{bmatrix} -e^t & e^{3t} & -e^{-2t} \\ 4e^t & 2e^{3t} & e^{-2t} \\ e^t & e^{3t} & e^{-2t} \end{bmatrix}$$

is a fundamental matrix solution of (3).

THEOREM: Let $\mathbf{X}(t)$ be a fundamental matrix solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Then,

$$e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0) \tag{4}$$

In other words, the product of any fundamental matrix solution of (1) with its inverse at $t = 0$ must yield e^{At}.

EXAMPLE: Find e^{At} if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution: The characteristic polynomial of the matrix A is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(5-\lambda)$$

so the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 3$, and $\lambda_3 = 5$.

(a) Let $\lambda = 1$. We use row operations:

$$\begin{bmatrix} 1-\lambda & 1 & 1 & 0 \\ 0 & 3-\lambda & 2 & 0 \\ 0 & 0 & 5-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$. Consequently,

$$\mathbf{x^1}(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is one solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

(b) Let $\lambda = 3$. We use row operations:

$$\begin{bmatrix} 1-\lambda & 1 & 1 & 0 \\ 0 & 3-\lambda & 2 & 0 \\ 0 & 0 & 5-\lambda & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_1 - \frac{1}{2}x_2 = 0 \\ x_3 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = \frac{1}{2}x_2 \\ x_3 = 0 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 3$. Consequently,

$$\mathbf{x^2}(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

is a second solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

(c) Let $\lambda = 5$. We use row operations:

$$\begin{bmatrix} 1-\lambda & 1 & 1 & 0 \\ 0 & 3-\lambda & 2 & 0 \\ 0 & 0 & 5-\lambda & 0 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/4 & -1/4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_1 - \dfrac{1}{2}x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \implies \begin{cases} x_1 = \dfrac{1}{2}x_3 \\ x_2 = x_3 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = \frac{1}{2}x_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$. Consequently,

$$\mathbf{x^3}(t) = e^{5t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

is a third solution of $\dot{\mathbf{x}} = A\mathbf{x}$. These solutions are clearly linearly independent. Therefore,

$$\mathbf{X}(t) = \begin{bmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}$$

is a fundamental matrix solution. Using the methods of Section 3.6, we compute

$$\mathbf{X}^{-1}(0) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Indeed, we have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

therefore

$$\exp\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} t\right) = \begin{bmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} e^t & -\frac{1}{2}e^t + \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t} + e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

REMARK: Note that

$$\begin{aligned} \text{Column 1} &= \quad 1 \cdot \mathbf{x^1}(t) + 0 \cdot \mathbf{x^2}(t) + 0 \cdot \mathbf{x^3}(t) \\ \text{Column 2} &= -\frac{1}{2} \cdot \mathbf{x^1}(t) + \frac{1}{2} \cdot \mathbf{x^2}(t) + 0 \cdot \mathbf{x^3}(t) \\ \text{Column 3} &= \quad 0 \cdot \mathbf{x^1}(t) - \frac{1}{2} \cdot \mathbf{x^2}(t) + \frac{1}{2} \cdot \mathbf{x^3}(t) \end{aligned}$$

In particular,

$$\exp\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}\right) = \begin{bmatrix} e & -\frac{1}{2}e + \frac{1}{2}e^3 & -\frac{1}{2}e^3 + \frac{1}{2}e^5 \\ 0 & e^3 & -e^3 + e^5 \\ 0 & 0 & e^5 \end{bmatrix}$$