Algebraic Properties of Solutions of Linear Systems

In this chapter we will consider simultaneous first-order differential equations in several variables, that is, equations of the form

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(t, x_1, \ldots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n) \end{cases} \tag{1}$$

In addition to equation (1), we will often impose initial conditions on the functions $x_1(t), \ldots, x_n(t)$. These will be of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0 \tag{1'}$$

Equation (1), together with the initial conditions (1′), is referred to as an initial-value problem.

First-order systems of differential equations also arise from higher-order equations for a single variable $y(t)$. Every nth-order differential equation for the single variable y can be converted into a system of n first-order equations for the variables

$$x_1(t) = y, \quad x_2(t) = \frac{dy}{dt}, \quad \ldots, \quad x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}$$

EXAMPLE: Convert the differential equation

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = 0$$

into a system of 2 first-order equations.

Solution: Let

$$x_1(t) = y \quad \text{and} \quad x_2(t) = \frac{dy}{dt}$$

From this it immediately follows that

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2(t) \quad \text{and} \quad \frac{dx_2}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

But the original equation implies

$$4\frac{d^2y}{dt^2} = -\frac{dy}{dt} - 3y$$

$$\frac{d^2y}{dt^2} = -\frac{\dfrac{dy}{dt} + 3y}{4}$$

therefore

$$\frac{dx_2}{dt} = -\frac{x_2 + 3x_1}{4}$$

So

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{x_2 + 3x_1}{4} \end{cases}$$

EXAMPLE: Convert the differential equation

$$a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_0y=0 \tag{2}$$

into a system of n first-order equations.

Solution: Let

$$\begin{aligned}
x_1(t) =& y\\
x_2(t) =& \frac{dy}{dt}=\frac{dx_1}{dt}\\
x_3(t) =& \frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{dx_2}{dt}\\
x_4(t) =& \frac{d^3y}{dt^3}=\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)=\frac{dx_3}{dt}\\
&\vdots\\
x_n(t) =& \frac{d^{n-1}y}{dt^{n-1}}=\frac{d}{dt}\left(\frac{d^{n-2}y}{dt^{n-2}}\right)=\frac{dx_{n-1}}{dt}\\
& \frac{d^ny}{dt^n}=\frac{d}{dt}\left(\frac{d^{n-1}y}{dt^{n-1}}\right)=\frac{dx_n}{dt}
\end{aligned}$$

From this and (2) it follows that

$$a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_0y=0$$

$$\begin{aligned}
a_n(t)\frac{d^ny}{dt^n} &= -a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}-a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}}-\ldots-a_0y\\
\frac{d^ny}{dt^n} &= -\frac{a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}}+\ldots+a_0y}{a_n(t)}\\
\frac{dx_n}{dt} &= -\frac{a_{n-1}(t)x_n+a_{n-2}(t)x_{n-1}+\ldots+a_0x_1}{a_n(t)}
\end{aligned}$$

So

$$\begin{cases}
\dfrac{dx_1}{dt}=x_2\\
\dfrac{dx_2}{dt}=x_3\\
\vdots\\
\dfrac{dx_{n-1}}{dt}=x_n\\
\dfrac{dx_n}{dt}=-\dfrac{a_{n-1}(t)x_n+a_{n-2}(t)x_{n-1}+\ldots+a_0x_1}{a_n(t)}
\end{cases}$$

EXAMPLE: Convert the initial-value problem

$$\frac{d^3y}{dt^3}+\left(\frac{dy}{dt}\right)^2+3y=e^t; \qquad y(0)=1, \quad y'(0)=0, \quad y''(0)=0 \tag{3}$$

into an initial-value problem for the variables $y,\ dy/dt$, and d^2y/dt^2.

Solution: Let

$$\begin{aligned} x_1(t) &= y \\ x_2(t) &= \frac{dy}{dt}=\frac{dx_1}{dt} \\ x_3(t) &= \frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{dx_2}{dt} \\ & \frac{d^3y}{dt^3}=\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)=\frac{dx_3}{dt} \end{aligned}$$

From this and (3) it follows that

$$\frac{d^3y}{dt^3}+\left(\frac{dy}{dt}\right)^2+3y=e^t$$

$$\frac{d^3y}{dt^3}=e^t-\left(\frac{dy}{dt}\right)^2-3y$$

$$\frac{dx_3}{dt}=e^t-x_2^2-3x_1$$

So

$$\begin{cases} \dfrac{dx_1}{dt}=x_2 \\ \dfrac{dx_2}{dt}=x_3 \\ \dfrac{dx_3}{dt}=e^t-x_2^2-3x_1 \end{cases}$$

Moreover, the functions x_1, x_2, and x_3 satisfy the initial conditions

$$\begin{aligned} x_1(0) &= y(0)=1 \\ x_2(0) &= y'(0)=0 \\ x_3(0) &= y''(0)=0 \end{aligned}$$

The most general system of n first-order linear equations has the form

$$\begin{cases} \dfrac{dx_1}{dt}=a_{11}(t)x_1+\ldots+a_{1n}(t)x_n+g_1(t) \\ \vdots \\ \dfrac{dx_n}{dt}=a_{n1}(t)x_1+\ldots+a_{nn}(t)x_n+g_n(t) \end{cases} \tag{4}$$

If each of the functions $g_1,\ldots g_n$ is identically zero, then the system (4) is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

Now, even the homogeneous linear system with constant coefficients

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + \ldots + a_{nn}x_n \end{cases} \tag{5}$$

is quite cumbersome to handle. This is especially true if n is large. Therefore, we seek to write these equations in as concise a manner as possible. To this end we introduce the concepts of *vectors* and *matrices*.

DEFINITION: A *vector*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a shorthand notation for the sequence of numbers $x_1, \ldots, x_n$. The numbers $x_1, \ldots, x_n$, are called the *components* of $\mathbf{x}$. If $x_1 = x_1(t), \ldots, x_n = x_n(t)$, then

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is called a vector-valued function. Its derivative $d\mathbf{x}(t)/dt$ is the vector-valued function

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \\ \vdots \\ \dfrac{dx_n(t)}{dt} \end{bmatrix}$$

DEFINITION: A *matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

is a shorthand notation for the array of numbers a_{ij} arranged in m rows and n columns.

DEFINITION: Let A be an $n \times n$ matrix with elements a_{ij} and let $\mathbf{x}$ be a vector with components $x_1, \dots, x_n$. We define the product of A with $\mathbf{x}$, denoted by $A\mathbf{x}$, as the vector whose ith component is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \qquad i = 1, 2, \ldots, n$$

In other words, the ith component of $A\mathbf{x}$ is the sum of the product of corresponding terms of the ith row of A with the vector $\mathbf{x}$. Thus,

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{bmatrix}$$

For example,

$$\begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}$$

Finally, we observe that the left-hand sides of (5)

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + \ldots + a_{1n}x_n \\ \quad\vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + \ldots + a_{nn}x_n \end{cases}$$

are the components of the vector dx/dt, while the right-hand sides of (5) are the components of the vector $A\mathbf{x}$. Hence, we can write (5) in the concise form

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{6}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Moreover, if $x_1(t), \ldots, x_n(t)$ satisfy the initial conditions

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0$$

then $\mathbf{x}(t)$ satisfies the initial-value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}^{\mathbf{0}}, \quad \text{where} \quad \mathbf{x}^{\mathbf{0}} = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix} \tag{7}$$

For example, the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - 7x_2 + 9x_3 \\ \dfrac{dx_2}{dt} = 15x_1 + x_2 - x_3 \\ \dfrac{dx_3}{dt} = 7x_1 + 6x_3 \end{cases}$$

can be written in the concise form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & -7 & 9 \\ 15 & 1 & -1 \\ 7 & 0 & 6 \end{bmatrix}$$

DEFINITION: Let c be a number and $\mathbf{x}$ a vector with n components $x_1, \ldots, x_n$. We define $c\mathbf{x}$ to be the vector whose components are $cx_1, \ldots, cx_n$, that is

$$c\mathbf{x} = c\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

DEFINITION: Let $\mathbf{x}$ and $\mathbf{y}$ be vectors with components $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. We define $\mathbf{x} + \mathbf{y}$ to be the vector whose components are $x_1 + y_1, \ldots, x_n + y_n$, that is

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

THEOREM 1: Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be two solutions of (6). Then

(a) $c\mathbf{x}(t)$ is a solution for any constant c

(b) $\mathbf{x}(t) + \mathbf{y}(t)$ is again a solution.

An immediate corollary of Theorem 1 is that any linear combination of solutions of (6) is again a solution of (6).

EXAMPLE: Solve the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -4x_1 \end{cases} \tag{8}$$

Solution: We first note that (8) can be rewritten in the following matrix form or

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

This system of equations can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} + 4y = 0 \tag{9}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{dy}{dt}$$

To find two linearly independent solutions of (9) we note that the auxiliary equation is

$$r^2 + 4 = 0$$

with the roots $r_{1,2} = \pm 2i$. Consequently,

$$y_1(t) = e^{0t}\cos 2t = \cos 2t \qquad \text{and} \qquad y_2(t) = e^{0t}\sin 2t = \sin 2t$$

are two solutions of (9). It follows that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$$

is a solution of (8). Hence by Theorem 1,

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} + c_2 \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix} \\ &= c_1 \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} \\ &= \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix} \end{aligned}$$

is a solution of (8) again.

REMARK: Another way to solve (8) will be discussed in Section 3.9.

EXAMPLE: Solve the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 \\ \dfrac{dx_2}{dt} = 2x_1 + x_2 \end{cases} \tag{10}$$

Solution 1: The system can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0 \tag{11}$$

by setting

$$x_2 = y \quad \text{and} \quad x_1 = \frac{1}{2}\left(\frac{dy}{dt} - y\right)$$

Indeed, if we rewrite the second equation of system (10) as

$$x_1 = \frac{1}{2}(x_2' - x_2)$$

and plug it into the first equation, we get

$$\frac{1}{2}(x_2' - x_2)' = \frac{1}{2}(x_2' - x_2)$$

$$x_2'' - x_2' = x_2' - x_2$$

$$x_2'' - 2x_2' + x_2 = 0$$

To find two linearly independent solutions of (11) we note that the auxiliary equation is

$$r^2 - 2r + 1 = 0$$

Thus, $r = 1$ is a repeated root. Consequently,

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = te^t$$

are two solutions of (11). It follows that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} (y_1'(t) - y_1(t))/2 \\ y_1(t) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (y_2'(t) - y_2(t))/2 \\ y_2(t) \end{pmatrix}$$

is a solution of (10). Hence by Theorem 1,

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} (y_1'(t) - y_1(t))/2 \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} (y_2'(t) - y_2(t))/2 \\ y_2(t) \end{pmatrix} \\ &= c_1 \begin{pmatrix} (e^t - e^t)/2 \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} (e^t + te^t - te^t)/2 \\ te^t \end{pmatrix} \\ &= c_1 \begin{pmatrix} 0 \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^t/2 \\ te^t \end{pmatrix} \\ &= \begin{pmatrix} c_2 e^t/2 \\ c_1 e^t + c_2 te^t \end{pmatrix} \end{aligned}$$

is a solution of (10) again.

Solution 2: We solve the first equation of system (10). We have

$$\frac{dx_1}{dt} = x_1$$

$$\frac{dx_1}{x_1} = dt$$

$$\int \frac{dx_1}{x_1} dt = \int dt$$

$$\ln|x_1| = t + c$$

$$e^{\ln|x_1|} = e^{t+c}$$

$$|x_1| = e^{t+c} = e^c e^t$$

so

$$x_1 = \pm e^c e^t$$

We can easily verify that the function $x_1 = 0$ is also a solution of $\frac{dx_1}{dt} = x_1$. So we can write the general solution in the form

$$x_1 = c_1 e^t$$

where c_1 is an arbitrary constant ($c_1 = e^c$, or $c_1 = -e^c$, or $c_1 = 0$). We now substitute $x_1 = c_1 e^t$ into the second equation of system (10):

$$x_2' = 2x_1 + x_2$$

$$x_2' = 2c_1 e^t + x_2$$

$$x_2' - x_2 = 2c_1 e^t$$

This is a first-order linear differential equation (see Section 1.2). Here $a(t) = -1$ so that

$$\mu(t) = \exp\left(\int a(t)dt\right) = \exp\left(-\int dt\right) = e^{-t}$$

Multiplying both sides of the equation $x_2' - x_2 = 2c_1 e^t$ by $\mu(t)$ we obtain the equivalent equation

$$e^{-t}\left(x_2' - x_2\right) = 2c_1 \qquad \text{or} \qquad \frac{d}{dt}\left(e^{-t}x_2\right) = 2c_1$$

Hence

$$e^{-t}x_2 = \int 2c_1 dt = 2c_1 t + c_2$$

so

$$x_2 = (2c_1 t + c_2)e^t$$

Therefore

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ (2c_1 t + c_2)e^t \end{pmatrix}$$

REMARK: Another way to solve (10) will be discussed in Section 3.10.

EXAMPLE: Solve the system

$$\begin{cases} x_1' = 7x_1 + 4x_2 \\ x_2' = -3x_1 - x_2 \end{cases} \tag{12}$$

Solution: The system can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 5y = 0 \tag{13}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{1}{4}\left(\frac{dy}{dt} - 7y\right)$$

Indeed, if we rewrite the first equation of system (12) as

$$x_2 = \frac{1}{4}\left(x_1' - 7x_1\right)$$

and plug it into the second equation, we get

$$\begin{aligned} \frac{1}{4}\left(x_1' - 7x_1\right)' &= -3x_1 - \frac{1}{4}\left(x_1' - 7x_1\right) \\ \left(x_1' - 7x_1\right)' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 7x_1' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 6x_1' + 5x_1 &= 0 \end{aligned}$$

To find two linearly independent solutions of (13) we note that the auxiliary equation is $r^2 - 6r + 5 = 0$ with the roots $r_1 = 1$ and $r_2 = 5$. Consequently,

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = e^{5t}$$

are two solutions of (13). It follows that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ (y_1'(t) - 7y_1(t))/4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_2(t) \\ (y_2'(t) - 7y_2(t))/4 \end{pmatrix}$$

is a solution of (12). Hence by Theorem 1,

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} y_1(t) \\ (y_1'(t) - 7y_1(t))/4 \end{pmatrix} + c_2 \begin{pmatrix} y_2(t) \\ (y_2'(t) - 7y_2(t))/4 \end{pmatrix} \\ &= c_1 \begin{pmatrix} e^t \\ (e^t - 7e^t)/4 \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ (5e^{5t} - 7e^{5t})/4 \end{pmatrix} \\ &= c_1 \begin{pmatrix} e^t \\ -3e^t/2 \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ -e^{5t}/2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{5t} \\ -3c_1 e^t/2 - c_2 e^{5t}/2 \end{pmatrix} \end{aligned}$$

is a solution of (12) again.

REMARK: In Section 3.4 we will show that the solutions that we found in the last three examples are the *general* solutions of the systems.