Algebraic Properties of Solutions of Linear Systems

In this chapter we will consider simultaneous first-order differential equations in several variables, that is, equations of the form

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(t, x_1, \ldots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n) \end{cases} \tag{1}$$

In addition to equation (1), we will often impose initial conditions on the functions $x_1(t), \ldots, x_n(t)$. These will be of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0 \tag{1'}$$

Equation (1), together with the initial conditions (1′), is referred to as an initial-value problem.

First-order systems of differential equations also arise from higher-order equations for a single variable $y(t)$. Every nth-order differential equation for the single variable y can be converted into a system of n first-order equations for the variables

$$x_1(t) = y, \quad x_2(t) = \frac{dy}{dt}, \quad \ldots, \quad x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}$$

EXAMPLE: Convert the differential equation

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = 0$$

into a system of 2 first-order equations.

Solution: Let

$$x_1(t) = y \quad \text{and} \quad x_2(t) = \frac{dy}{dt}$$

From this it immediately follows that

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2(t) \quad \text{and} \quad \frac{dx_2}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

But the original equation implies

$$4\frac{d^2y}{dt^2} = -\frac{dy}{dt} - 3y$$

$$\frac{d^2y}{dt^2} = -\frac{\dfrac{dy}{dt} + 3y}{4}$$

therefore

$$\frac{dx_2}{dt} = -\frac{x_2 + 3x_1}{4}$$

So

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{x_2 + 3x_1}{4} \end{cases}$$

EXAMPLE: Convert the differential equation

$$a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_0y=0 \tag{2}$$

into a system of n first-order equations.

Solution: Let

$$\begin{aligned}
x_1(t)&=y\\
x_2(t)&=\frac{dy}{dt}=\frac{dx_1}{dt}\\
x_3(t)&=\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{dx_2}{dt}\\
x_4(t)&=\frac{d^3y}{dt^3}=\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)=\frac{dx_3}{dt}\\
&\vdots\\
x_n(t)&=\frac{d^{n-1}y}{dt^{n-1}}=\frac{d}{dt}\left(\frac{d^{n-2}y}{dt^{n-2}}\right)=\frac{dx_{n-1}}{dt}\\
&\frac{d^ny}{dt^n}=\frac{d}{dt}\left(\frac{d^{n-1}y}{dt^{n-1}}\right)=\frac{dx_n}{dt}
\end{aligned}$$

From this and (2) it follows that

$$\begin{aligned}
a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_0y=0\\
a_n(t)\frac{d^ny}{dt^n}&=-a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}-a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}}-\ldots-a_0y\\
\frac{d^ny}{dt^n}&=-\frac{a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}}+\ldots+a_0y}{a_n(t)}\\
\frac{dx_n}{dt}&=-\frac{a_{n-1}(t)x_n+a_{n-2}(t)x_{n-1}+\ldots+a_0x_1}{a_n(t)}
\end{aligned}$$

So

$$\begin{cases}
\dfrac{dx_1}{dt}=x_2\\
\dfrac{dx_2}{dt}=x_3\\
\vdots\\
\dfrac{dx_{n-1}}{dt}=x_n\\
\dfrac{dx_n}{dt}=-\dfrac{a_{n-1}(t)x_n+a_{n-2}(t)x_{n-1}+\ldots+a_0x_1}{a_n(t)}
\end{cases}$$

EXAMPLE: Convert the initial-value problem

$$\frac{d^3y}{dt^3} + \left(\frac{dy}{dt}\right)^2 + 3y = e^t; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \tag{3}$$

into an initial-value problem for the variables y, dy/dt, and d^2y/dt^2.

Solution: Let

$$\begin{aligned} x_1(t) =& y \\ x_2(t) =& \frac{dy}{dt} = \frac{dx_1}{dt} \\ x_3(t) =& \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx_2}{dt} \\ & \frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) = \frac{dx_3}{dt} \end{aligned}$$

From this and (3) it follows that

$$\frac{d^3y}{dt^3} + \left(\frac{dy}{dt}\right)^2 + 3y = e^t$$

$$\frac{d^3y}{dt^3} = e^t - \left(\frac{dy}{dt}\right)^2 - 3y$$

$$\frac{dx_3}{dt} = e^t - x_2^2 - 3x_1$$

So

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dfrac{dx_3}{dt} = e^t - x_2^2 - 3x_1 \end{cases}$$

Moreover, the functions x_1, x_2, and x_3 satisfy the initial conditions

$$x_1(0) = y(0) = 1$$

$$x_2(0) = y'(0) = 0$$

$$x_3(0) = y''(0) = 0$$

The most general system of n first-order linear equations has the form

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n + g_1(t) \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n + g_n(t) \end{cases} \tag{4}$$

If each of the functions $g_1, \ldots g_n$ is identically zero, then the system (4) is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

Now, even the homogeneous linear system with constant coefficients

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + \ldots + a_{nn}x_n \end{cases} \tag{5}$$

is quite cumbersome to handle. This is especially true if n is large. Therefore, we seek to write these equations in as concise a manner as possible. To this end we introduce the concepts of *vectors* and *matrices*.

DEFINITION: A *vector*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a shorthand notation for the sequence of numbers $x_1, \ldots, x_n$. The numbers $x_1, \ldots, x_n$, are called the *components* of $\mathbf{x}$. If $x_1 = x_1(t), \ldots, x_n = x_n(t)$, then

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is called a vector-valued function. Its derivative $d\mathbf{x}(t)/dt$ is the vector-valued function

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \\ \vdots \\ \dfrac{dx_n(t)}{dt} \end{bmatrix}$$

DEFINITION: A *matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

is a shorthand notation for the array of numbers a_{ij} arranged in m rows and n columns.

DEFINITION: Let A be an $n \times n$ matrix with elements a_{ij} and let $\mathbf{x}$ be a vector with components $x_1, \ldots, x_n$. We define the product of A with $\mathbf{x}$, denoted by $A\mathbf{x}$, as the vector whose ith component is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \qquad i = 1, 2, \ldots, n$$

In other words, the ith component of $A\mathbf{x}$ is the sum of the product of corresponding terms of the ith row of A with the vector $\mathbf{x}$. Thus,

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{bmatrix}$$

For example,

$$\begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}$$

Finally, we observe that the left-hand sides of (5)

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + \ldots + a_{1n}x_n \\ \quad\vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + \ldots + a_{nn}x_n \end{cases}$$

are the components of the vector dx/dt, while the right-hand sides of (5) are the components of the vector $A\mathbf{x}$. Hence, we can write (5) in the concise form

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{6}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Moreover, if $x_1(t), \ldots, x_n(t)$ satisfy the initial conditions

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0$$

then $\mathbf{x}(t)$ satisfies the initial-value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x^0}, \quad \text{where} \quad \mathbf{x^0} = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix} \tag{7}$$

For example, the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - 7x_2 + 9x_3 \\ \dfrac{dx_2}{dt} = 15x_1 + x_2 - x_3 \\ \dfrac{dx_3}{dt} = 7x_1 + 6x_3 \end{cases}$$

can be written in the concise form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & -7 & 9 \\ 15 & 1 & -1 \\ 7 & 0 & 6 \end{bmatrix}$$

DEFINITION: Let c be a number and $\mathbf{x}$ a vector with n components $x_1, \ldots, x_n$. We define $c\mathbf{x}$ to be the vector whose components are $cx_1, \ldots, cx_n$, that is

$$c\mathbf{x} = c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

DEFINITION: Let $\mathbf{x}$ and $\mathbf{y}$ be vectors with components $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. We define $\mathbf{x} + \mathbf{y}$ to be the vector whose components are $x_1 + y_1, \ldots, x_n + y_n$, that is

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

THEOREM 1: Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be two solutions of (6). Then

(a) $c\mathbf{x}(t)$ is a solution for any constant c

(b) $\mathbf{x}(t) + \mathbf{y}(t)$ is again a solution.

An immediate corollary of Theorem 1 is that any linear combination of solutions of (6) is again a solution of (6).

EXAMPLE: Solve the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -4x_1 \end{cases} \tag{8}$$

Solution: We first note that (8) can be rewritten in the following matrix form or

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

This system of equations can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} + 4y = 0 \tag{9}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{dy}{dt}$$

To find two linearly independent solutions of (9) we note that the characteristic equation is

$$r^2 + 4 = 0$$

with the roots $r_{1,2} = \pm 2i$. Consequently,

$$y_1(t) = e^{0t}\cos 2t = \cos 2t \qquad \text{and} \qquad y_2(t) = e^{0t}\sin 2t = \sin 2t$$

are two solutions of (9). It follows that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$$

is a solution of (8). Hence by Theorem 1,

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} + c_2 \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix} \\ &= c_1 \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} \\ &= \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix} \end{aligned}$$

is a solution of (8) again.

REMARK: Another way to solve (8) will be discussed in Section 3.9.

EXAMPLE: Solve the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 \\ \dfrac{dx_2}{dt} = 2x_1 + x_2 \end{cases} \tag{10}$$

Solution 1: The system can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0 \tag{11}$$

by setting

$$x_2 = y \quad \text{and} \quad x_1 = \frac{1}{2}\left(\frac{dy}{dt} - y\right)$$

Indeed, if we rewrite the second equation of system (10) as

$$x_1 = \frac{1}{2}\left(x_2' - x_2\right)$$

and plug it into the first equation, we get

$$\frac{1}{2}\left(x_2' - x_2\right)' = \frac{1}{2}\left(x_2' - x_2\right)$$

$$x_2'' - x_2' = x_2' - x_2$$

$$x_2'' - 2x_2' + x_2 = 0$$

To find two linearly independent solutions of (11) we note that the characteristic equation is

$$r^2 - 2r + 1 = 0$$

Thus, $r = 1$ is a repeated root. Consequently,

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = te^t$$

are two solutions of (11). It follows that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} (y_1'(t) - y_1(t))/2 \\ y_1(t) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (y_2'(t) - y_2(t))/2 \\ y_2(t) \end{pmatrix}$$

is a solution of (10). Hence by Theorem 1,

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} (y_1'(t) - y_1(t))/2 \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} (y_2'(t) - y_2(t))/2 \\ y_2(t) \end{pmatrix} \\ &= c_1 \begin{pmatrix} (e^t - e^t)/2 \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} (e^t + te^t - te^t)/2 \\ te^t \end{pmatrix} \\ &= c_1 \begin{pmatrix} 0 \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^t/2 \\ te^t \end{pmatrix} \\ &= \begin{pmatrix} c_2 e^t/2 \\ c_1 e^t + c_2 te^t \end{pmatrix} \end{aligned}$$

is a solution of (10) again.

Solution 2: We solve the first equation of system (10). We have

$$\frac{dx_1}{dt} = x_1$$

$$\frac{dx_1}{x_1} = dt$$

$$\int \frac{dx_1}{x_1} dt = \int dt$$

$$\ln |x_1| = t + c$$

$$e^{\ln |x_1|} = e^{t+c}$$

$$|x_1| = e^{t+c} = e^c e^t$$

so

$$x_1 = \pm e^c e^t$$

We can easily verify that the function $x_1 = 0$ is also a solution of $\frac{dx_1}{dt} = x_1$. So we can write the general solution in the form

$$x_1 = c_1 e^t$$

where c_1 is an arbitrary constant ($c_1 = e^c$, or $c_1 = -e^c$, or $c_1 = 0$). We now substitute $x_1 = c_1 e^t$ into the second equation of system (10):

$$x_2' = 2x_1 + x_2$$

$$x_2' = 2c_1 e^t + x_2$$

$$x_2' - x_2 = 2c_1 e^t$$

This is a first-order linear differential equation (see Section 1.2). Here $a(t) = -1$ so that

$$\mu(t) = \exp\left(\int a(t) dt\right) = \exp\left(-\int dt\right) = e^{-t}$$

Multiplying both sides of the equation $x_2' - x_2 = 2c_1 e^t$ by $\mu(t)$ we obtain the equivalent equation

$$e^{-t}\left(x_2' - x_2\right) = 2c_1 \qquad \text{or} \qquad \frac{d}{dt}\left(e^{-t} x_2\right) = 2c_1$$

Hence

$$e^{-t} x_2 = \int 2c_1 dt = 2c_1 t + c_2$$

so

$$x_2 = (2c_1 t + c_2)e^t$$

Therefore

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ (2c_1 t + c_2)e^t \end{pmatrix}$$

REMARK: Another way to solve (10) will be discussed in Section 3.10.

EXAMPLE: Solve the system

$$\begin{cases} x_1' = 7x_1 + 4x_2 \\ x_2' = -3x_1 - x_2 \end{cases} \tag{12}$$

Solution: The system can be derived from the second-order differential equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 5y = 0 \tag{13}$$

by setting

$$x_1 = y \quad \text{and} \quad x_2 = \frac{1}{4}\left(\frac{dy}{dt} - 7y\right)$$

Indeed, if we rewrite the first equation of system (12) as

$$x_2 = \frac{1}{4}(x_1' - 7x_1)$$

and plug it into the second equation, we get

$$\begin{aligned} \frac{1}{4}(x_1' - 7x_1)' &= -3x_1 - \frac{1}{4}(x_1' - 7x_1) \\ (x_1' - 7x_1)' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 7x_1' &= -12x_1 - x_1' + 7x_1 \\ x_1'' - 6x_1' + 5x_1 &= 0 \end{aligned}$$

To find two linearly independent solutions of (13) we note that the characteristic equation is $r^2 - 6r + 5 = 0$ with the roots $r_1 = 1$ and $r_2 = 5$. Consequently,

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = e^{5t}$$

are two solutions of (13). It follows that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ (y_1'(t) - 7y_1(t))/4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_2(t) \\ (y_2'(t) - 7y_2(t))/4 \end{pmatrix}$$

is a solution of (12). Hence by Theorem 1,

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} y_1(t) \\ (y_1'(t) - 7y_1(t))/4 \end{pmatrix} + c_2 \begin{pmatrix} y_2(t) \\ (y_2'(t) - 7y_2(t))/4 \end{pmatrix} \\ &= c_1 \begin{pmatrix} e^t \\ (e^t - 7e^t)/4 \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ (5e^{5t} - 7e^{5t})/4 \end{pmatrix} \\ &= c_1 \begin{pmatrix} e^t \\ -3e^t/2 \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ -e^{5t}/2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{5t} \\ -3c_1 e^t/2 - c_2 e^{5t}/2 \end{pmatrix} \end{aligned}$$

is a solution of (12) again.

REMARK: Another way to solve (12) will be discussed in Section 3.8. Also, in Section 3.4 we will show that the solutions that we found in the last three examples are the *general* solutions of the systems.