Series Solutions

We return now to the general homogeneous linear second-order equation

$$L[y] = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \tag{1}$$

with $P(t)$ unequal to zero in the interval $\alpha < t < \beta$. It was shown in Section 2.1 that every solution $y(t)$ of (1) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

where $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of (1). Thus, the problem of finding all solutions of (1) is reduced to the simpler problem of finding just two solutions. In Section 2.2 we handled the special case where P, Q, and R are constants. The next simplest case is when $P(t)$, $Q(t)$, and $R(t)$ are polynomials in t.

EXAMPLE: Find two linearly independent solutions of the equation

$$L[y] = \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} - 2y = 0 \tag{2}$$

Solution: We set

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots = \sum_{n=0}^{\infty} a_n t^n$$

Computing

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \ldots = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

and

$$\frac{d^2y}{dt^2} = 2a_2 + 6a_3 t + 12a_4 t^2 + \ldots = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

we see that $y(t)$ is a solution of (2) if

$$\begin{aligned} L[y](t) &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 2t\sum_{n=0}^{\infty} n a_n t^{n-1} - 2\sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 2\sum_{n=0}^{\infty} n a_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0 \end{aligned} \tag{3}$$

Our next step is to rewrite the first summation in (3) so that the exponent of the general term is n, instead of $n-2$. We have

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} &= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} t^n \\ &= (-2+2)(-2+1) a_{-2+2} t^{-2} + (-1+2)(-1+1) a_{-1+2} t^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n \\ &= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n \end{aligned}$$

therefore we can rewrite (3) in the form

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty} na_nt^n - 2\sum_{n=0}^{\infty} a_nt^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2na_n - 2a_n]t^n = 0 \qquad (4)$$

Therefore the coefficients must satisfy

$$(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0$$

That is,

$$a_{n+2} = \frac{2na_n + 2a_n}{(n+2)(n+1)} = \frac{2(n+1)a_n}{(n+2)(n+1)} = \frac{2a_n}{n+2} \qquad (5)$$

To find two solutions of (2), we choose two different sets of values of a_0, and a_1. The simplest possible choices are (i) $a_0 = 1$, $a_1 = 0$; (ii) $a_0 = 0$, $a_1 = 1$.

(i) $a_0 = 1$, $a_1 = 0$.

In this case, all the odd coefficients $a_1, a_3, a_5, \ldots$ are zero since

$$a_1 = 0, \qquad a_3 = \frac{2a_1}{3} = \frac{2\cdot 0}{3} = 0, \qquad a_5 = \frac{2a_3}{5} = \frac{2\cdot 0}{5} = 0$$

and so on. The even coefficients are determined from the relations

$$a_2 = \frac{2a_0}{2} = a_0 = 1, \qquad a_4 = \frac{2a_2}{4} = \frac{a_2}{2} = \frac{1}{2}, \qquad a_6 = \frac{2a_4}{6} = \frac{a_4}{3} = \frac{1}{2\cdot 3}$$

and so on. Proceeding inductively, we find that

$$a_{2n} = \frac{1}{2\cdot 3\ldots n} = \frac{1}{n!}$$

Hence,

$$y_1(t) = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \ldots = e^{t^2}$$

is one solution of (2).

(ii) $a_0 = 0$, $a_1 = 1$.

In this case, all the even coefficients $a_0, a_2, a_4, \ldots$ are zero since

$$a_0 = 0, \qquad a_2 = \frac{2a_0}{2} = \frac{2\cdot 0}{2} = 0, \qquad a_4 = \frac{2a_2}{4} = \frac{2\cdot 0}{4} = 0$$

and so on. The odd coefficients are determined from the relations

$$a_1 = 1, \qquad a_3 = \frac{2a_1}{3} = \frac{2}{3}, \qquad a_5 = \frac{2a_3}{5} = \frac{2}{5}\cdot\frac{2}{3} \qquad a_7 = \frac{2a_5}{7} = \frac{2}{7}\cdot\frac{2}{5}\cdot\frac{2}{3}$$

and so on. Proceeding inductively, we find that

$$a_{2n+1} = \frac{2^n}{3\cdot 5\ldots(2n+1)}$$

Hence,

$$y_2(t) = t + \frac{2t^3}{3} + \frac{2^2t^5}{3\cdot 5} + \ldots = \sum_{n=0}^{\infty}\frac{2^nt^{2n+1}}{3\cdot 5\ldots(2n+1)}$$

is a second solution of (2).

EXAMPLE: Find the general solution of the equation

$$y'' + ty = 0 \tag{6}$$

Solution: We set

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots = \sum_{n=0}^{\infty} a_n t^n$$

We have (see the example above)

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} n a_n t^{n-1} \qquad \text{and} \qquad \frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

therefore $y(t)$ is a solution of (6) if

$$\begin{aligned} y'' + ty &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + t \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+1} = 0 \end{aligned} \tag{7}$$

Our next step is to rewrite the first summation in (7) so that the exponent of the general term is $n+1$, instead of $n-2$. We have

$$\begin{aligned} &\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} \\ &= \sum_{n=-3}^{\infty} (n+3)(n+2) a_{n+3} t^{n+1} \\ &= (-3+3)(-3+2) a_{-3+3} t^{-3+1} + (-2+3)(-2+2) a_{-2+3} t^{-2+1} + (-1+3)(-1+2) a_{-1+3} t^{-1+1} \\ &\quad + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} t^{n+1} \\ &= 0 + 0 + 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} t^{n+1} \end{aligned}$$

therefore we can rewrite (7) in the form

$$2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+1} = 2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) a_{n+3} + a_n] t^{n+1} = 0 \tag{8}$$

Therefore the coefficients must satisfy

$$a_2 = 0 \quad \text{and} \quad (n+3)(n+2) a_{n+3} + a_n = 0$$

That is,

$$a_2 = 0 \quad \text{and} \quad a_{n+3} = -\frac{a_n}{(n+3)(n+2)} \tag{9}$$

To find two solutions of (6), we choose two different sets of values of a_0 and a_1 (keeping in mind that $a_2 = 0$). The simplest possible choices are (i) $a_0 = 1$, $a_1 = 0$, $a_2 = 0$; (ii) $a_0 = 0$, $a_1 = 1$, $a_2 = 0$.

(i) $a_0 = 1$, $a_1 = 0$, $a_2 = 0$.

In this case, $a_1, a_4, a_7, \dots$ and $a_2, a_5, a_8, \dots$ are zero since

$$a_1 = 0, \qquad a_4 = -\frac{a_1}{4\cdot 3} = -\frac{0}{4\cdot 3} = 0, \qquad a_7 = -\frac{a_4}{7\cdot 6} = -\frac{0}{7\cdot 6} = 0$$

and

$$a_2 = 0, \qquad a_5 = -\frac{a_2}{5\cdot 4} = -\frac{0}{5\cdot 4} = 0, \qquad a_8 = -\frac{a_5}{8\cdot 7} = -\frac{0}{8\cdot 7} = 0$$

and so on. The coefficients $a_0, a_3, a_6, \dots$ are determined from the relations

$$\begin{aligned}
a_0 &= 1 \\
a_3 &= -\frac{a_0}{3\cdot 2} = -\frac{1}{3\cdot 2} = -\frac{1}{3!} \\
a_6 &= -\frac{a_3}{6\cdot 5} = \frac{1}{3!\cdot 6\cdot 5} = \frac{1\cdot 4}{3!\cdot 6\cdot 5\cdot 4} = \frac{1\cdot 4}{6!} \\
a_9 &= -\frac{a_6}{9\cdot 8} = -\frac{1\cdot 4}{6!\cdot 9\cdot 8} = -\frac{1\cdot 4\cdot 7}{6!\cdot 9\cdot 8\cdot 7} = -\frac{1\cdot 4\cdot 7}{9!} \\
a_{12} &= -\frac{a_9}{12\cdot 11} = \frac{1\cdot 4\cdot 7}{9!\cdot 12\cdot 11} = \frac{1\cdot 4\cdot 7\cdot 10}{9!\cdot 12\cdot 11\cdot 10} = \frac{1\cdot 4\cdot 7\cdot 10}{12!}
\end{aligned}$$

and so on. Proceeding inductively, we find that

$$a_{3n} = (-1)^n \frac{1\cdot 4\cdot 7 \dots (3n-2)}{(3n)!}$$

Hence,

$$\begin{aligned}
y_1(t) &= 1 - \frac{1}{3!}t^3 + \frac{1\cdot 4}{6!}t^6 - \frac{1\cdot 4\cdot 7}{9!}t^9 + \frac{1\cdot 4\cdot 7\cdot 10}{12!}t^{12} + \dots \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 4\cdot 7 \dots (3n-2)}{(3n)!} t^{3n}
\end{aligned}$$

is one solution of (6).

(ii) $a_0 = 0$, $a_1 = 1$, $a_2 = 0$.

In this case, $a_0, a_3, a_6, \dots$ and $a_2, a_5, a_8, \dots$ are zero since

$$a_0 = 0, \qquad a_3 = -\frac{a_0}{3\cdot 2} = -\frac{0}{3\cdot 2} = 0, \qquad a_6 = -\frac{a_3}{6\cdot 5} = -\frac{0}{6\cdot 5} = 0$$

and

$$a_2 = 0, \qquad a_5 = -\frac{a_2}{5\cdot 4} = -\frac{0}{5\cdot 4} = 0, \qquad a_8 = -\frac{a_5}{8\cdot 7} = -\frac{0}{8\cdot 7} = 0$$

and so on.

The coefficients $a_1, a_4, a_7, \dots$ are determined from the relations

$$\begin{aligned}
a_1 &= 1 \\
a_4 &= -\frac{a_1}{4\cdot 3} = -\frac{1}{4\cdot 3} = -\frac{2}{4\cdot 3\cdot 2} = -\frac{2}{4!} \\
a_7 &= -\frac{a_4}{7\cdot 6} = \frac{2}{4!\cdot 7\cdot 6} = \frac{2\cdot 5}{4!\cdot 7\cdot 6\cdot 5} = \frac{2\cdot 5}{7!} \\
a_{10} &= -\frac{a_7}{10\cdot 9} = -\frac{2\cdot 5}{7!\cdot 10\cdot 9} = -\frac{2\cdot 5\cdot 8}{7!\cdot 10\cdot 9\cdot 8} = -\frac{2\cdot 5\cdot 8}{10!} \\
a_{13} &= -\frac{a_{10}}{13\cdot 12} = \frac{2\cdot 5\cdot 8}{10!\cdot 13\cdot 12} = \frac{2\cdot 5\cdot 8\cdot 11}{10!\cdot 13\cdot 12\cdot 11} = \frac{2\cdot 5\cdot 8\cdot 11}{13!}
\end{aligned}$$

and so on. Proceeding inductively, we find that

$$a_{3n+1} = (-1)^n \frac{2\cdot 5\cdot 8 \dots (3n-1)}{(3n+1)!}$$

Hence,

$$y_2(t) = t - \frac{2}{4!}t^4 + \frac{2\cdot 5}{7!}t^7 - \frac{2\cdot 5\cdot 8}{10!}t^{10} + \frac{2\cdot 5\cdot 8\cdot 11}{13!}t^{13} + \dots = t + \sum_{n=1}^{\infty} (-1)^n \frac{2\cdot 5\cdot 8 \dots (3n-1)}{(3n+1)!} t^{3n+1}$$

is a second solution of (6).

Therefore the general solution of (6) is

$$\begin{aligned}
y(t) &= c_1 y_1(t) + c_2 y_2(t) \\
&= c_1 \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 4\cdot 7 \dots (3n-2)}{(3n)!} t^{3n}\right) + c_2 \left(t + \sum_{n=1}^{\infty} (-1)^n \frac{2\cdot 5\cdot 8 \dots (3n-1)}{(3n+1)!} t^{3n+1}\right)
\end{aligned}$$

THEOREM 6: Let the variable t assume complex values, and let z_0, be the point closest to t_0 at which f or one of its derivatives fails to exist. Compute the distance ρ, in the complex plane, between t_0 and z_0. Then, the Taylor series of f about t_0 converges for $|t - t_0| < \rho$, and diverges for $|t - t_0| > \rho$.

THEOREM 7: Let the functions $Q(t)/P(t)$ and $R(t)/P(t)$ have convergent Taylor series expansions about $t = t_0$, for $|t - t_0| < \rho$. Then, every solution $y(t)$ of the differential equation

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \tag{10}$$

is analytic at $t = t_0$, and the radius of convergence of its Taylor series expansion about $t = t_0$ is at least ρ. The coefficients $a_2, a_3, \dots,$ in the Taylor series expansion

$$y(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \dots \tag{11}$$

are determined by plugging the series (11) into the differential equation (10) and setting the sum of the coefficients of like powers of t in this expression equal to zero.

EXAMPLE:

(a) Find two linearly independent solutions of the equation

$$L[y]=\frac{d^2y}{dt^2}+\frac{3t}{1+t^2}\frac{dy}{dt}+\frac{1}{1+t^2}y=0 \tag{12}$$

(b) Find the solution $y(t)$ of (12) which satisfies the initial conditions $y(0)=2$, $y'(0)=3$.

Solution: We first multiply both sides of (12) by $1+t^2$ to obtain the equivalent equation

$$L[y]=(1+t^2)\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+y=0 \tag{13}$$

and set

$$y(t)=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+\ldots=\sum_{n=0}^{\infty}a_nt^n$$

We have (see the example above)

$$\frac{dy}{dt}=\sum_{n=0}^{\infty}na_nt^{n-1} \qquad \text{and} \qquad \frac{d^2y}{dt^2}=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}$$

therefore $y(t)$ is a solution of (13) if

$$\begin{aligned}
L[y](t)&=(1+t^2)\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+3t\sum_{n=0}^{\infty}na_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^n\\
&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+t^2\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+3t\sum_{n=0}^{\infty}na_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^n\\
&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}t^2+\sum_{n=0}^{\infty}3na_nt^{n-1}t+\sum_{n=0}^{\infty}a_nt^n\\
&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}n(n-1)a_nt^n+\sum_{n=0}^{\infty}3na_nt^n+\sum_{n=0}^{\infty}a_nt^n\\
&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}[n(n-1)+3n+1]a_nt^n=0
\end{aligned}$$

Note that

$$n(n-1)+3n+1=n^2-n+3n+1=n^2+2n+1=(n+1)^2$$

therefore

$$L[y](t)=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}(n+1)^2a_nt^n=0 \tag{14}$$

In the first example it was shown that

$$\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$$

therefore we can rewrite (14) in the form

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)^2 a_n t^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)^2 a_n]t^n = 0 \qquad (15)$$

Therefore the coefficients must satisfy

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_n = 0$$

That is,

$$a_{n+2} = -\frac{(n+1)^2 a_n}{(n+2)(n+1)} = -\frac{(n+1)a_n}{n+2} \qquad (16)$$

To find two solutions of (12), we choose two different sets of values of a_0, and a_1. The simplest possible choices are (i) $a_0 = 1$, $a_1 = 0$; (ii) $a_0 = 0$, $a_1 = 1$.

(i) $a_0 = 1$, $a_1 = 0$.

In this case, all the odd coefficients $a_1, a_3, a_5, \ldots$ are zero since

$$a_1 = 0, \qquad a_3 = -\frac{2a_1}{3} = -\frac{2\cdot 0}{3} = 0, \qquad a_5 = -\frac{4a_3}{5} = -\frac{4\cdot 0}{5} = 0$$

and so on. The even coefficients are determined from the relations

$$a_0 = 1, \qquad a_2 = -\frac{a_0}{2} = -\frac{1}{2}, \qquad a_4 = -\frac{3a_2}{4} = \frac{1\cdot 3}{2\cdot 4}, \qquad a_6 = -\frac{5a_4}{6} = -\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$$

and so on. Proceeding inductively, we find that

$$a_{2n} = (-1)^n \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} = (-1)^n \frac{1\cdot 3\ldots(2n-1)}{2^n n!}$$

Thus,

$$y_1(t) = 1 - \frac{t^2}{2} + \frac{1\cdot 3}{2\cdot 4}t^4 + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{1\cdot 3\ldots(2n-1)}{2^n n!}t^{2n} \qquad (17)$$

is one solution of (12). The ratio of the $(n+1)$st term to the nth term of $y_1(t)$ is

$$-\frac{1\cdot 3\ldots(2n-1)(2n+1)t^{2n+2}}{2^{n+1}(n+1)!} \times \frac{2^n n!}{1\cdot 3\ldots(2n-1)t^{2n}} = -\frac{(2n+1)t^2}{2(n+1)}$$

and the absolute value of this quantity approaches t^2 as n approaches infinity. Hence, by the Cauchy ratio test, the infinite series (17) converges for $|t| < 1$, and diverges for $|t| > 1$.

(ii) $a_0 = 0$, $a_1 = 1$.

In this case, all the even coefficients $a_0, a_2, a_4, \ldots$ are zero since

$$a_0 = 0, \qquad a_2 = -\frac{a_0}{2} = -\frac{0}{2} = 0, \qquad a_4 = -\frac{3a_2}{4} = -\frac{3\cdot 0}{4} = 0$$

and so on. The odd coefficients are determined from the relations

$$a_1 = 1, \qquad a_3 = -\frac{2a_1}{3} = -\frac{2}{3}, \qquad a_5 = -\frac{4a_3}{5} = \frac{2\cdot 4}{3\cdot 5} \qquad a_7 = -\frac{6a_5}{7} = -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

and so on. Proceeding inductively, we find that

$$a_{2n+1} = (-1)^n \frac{2 \cdot 4 \ldots 2n}{3 \cdot 5 \ldots (2n+1)} = (-1)^n \frac{2^n n!}{3 \cdot 5 \ldots (2n+1)}$$

Thus,

$$y_2(t) = t - \frac{2}{3}t^3 + \frac{2 \cdot 4}{3 \cdot 5}t^5 + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{2^n n!}{3 \cdot 5 \ldots (2n+1)} t^{2n+1}$$

is one solution of (12). The ratio of the $(n+1)$st term to the nth term of $y_2(t)$ is

$$-\frac{2^{n+1}(n+1)!t^{2n+3}}{3 \cdot 5 \ldots (2n+1)(2n+3)} \times \frac{3 \cdot 5 \ldots (2n+1)}{2^n n! t^{2n+1}} = -\frac{2(n+1)t^2}{2n+3}$$

and the absolute value of this quantity approaches t^2 as n approaches infinity. Hence, by the Cauchy ratio test, the infinite series (17) converges for $|t| < 1$, and diverges for $|t| > 1$.

(b) The solution $y_1(t)$ satisfies the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$, while $y_2(t)$ satisfies the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$. Hence

$$y(t) = 2y_1(t) + 3y_2(t)$$

satisfies the initial conditions $y(0) = 2$, $y'(0) = 3$.

Singular points, Euler equations

The differential equation

$$L[y] = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \tag{18}$$

is said to be singular at $t = t_0$ if $P(t_0) = 0$. Solutions of (18) frequently become very large, or oscillate very rapidly, in a neighborhood of the singular point t_0. Thus, solutions of (18) may not even be continuous, let along analytic at t_0, and the method of power series solution will fail to work, in general.

Our goal is to find a class of singular equations which we can solve for t near t_0. To this end we will first study a very simple equation, known as Euler's equation, which is singular, but easily solvable.

DEFINITION: The differential equation

$$L[y] = t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0 \tag{19}$$

where α and β are constants is known as Euler's equation.

We will assume, for simplicity, that $t > 0$.

Case I: $(\alpha - 1)^2 - 4\beta > 0$. In this case the general solution of (19) is

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}$$

where r_1 and r_2 are roots of the equation

$$r^2 + (\alpha - 1)r + \beta = 0$$

EXAMPLE: Find the general solution of

$$L[y] = t^2\frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 0, \quad t > 0$$

Solution: Here $\alpha = 4$ and $\beta = 2$. One can check that $(\alpha - 1)^2 - 4\beta > 0$, therefore the general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{-2}$$

since -1 and -2 are roots of the equation

$$r^2 + 3r + 2 = 0$$

Case II: $(\alpha - 1)^2 - 4\beta = 0$. In this case the general solution of (19) is

$$y(t) = (c_1 + c_2 \ln t)t^r, \quad t > 0$$

where

$$r = \frac{1-\alpha}{2}$$

EXAMPLE: Find the general solution of

$$L[y] = t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0, \quad t > 0$$

Solution: Here $\alpha = -5$ and $\beta = 9$. One can check that $(\alpha - 1)^2 - 4\beta = 0$, therefore the general solution is

$$y(t) = (c_1 + c_2 \ln t)t^{(1+5)/2} = (c_1 + c_2 \ln t)t^3, \quad t > 0$$

Case III: $(\alpha-1)^2-4\beta<0$. In this case the general solution is

$$y(t)=t^{\lambda}\Big[c_1\cos(\mu\ln t)+c_2\sin(\mu\ln t)\Big]$$

where

$$\lambda=\frac{1-\alpha}{2},\qquad \mu=\frac{\left[4\beta-(\alpha-1)^2\right]^{1/2}}{2}$$

EXAMPLE: Find the general solution of

$$L[y]=t^2\frac{d^2y}{dt^2}-5t\frac{dy}{dt}+25y=0,\quad t>0$$

Solution: Here $\alpha=-5$ and $\beta=25$. One can check that $(\alpha-1)^2-4\beta<0$. We have

$$\lambda=\frac{1+5}{2}=3,\qquad \mu=\frac{\left[4\cdot 25-(-5-1)^2\right]^{1/2}}{2}=4$$

therefore the general solution is

$$y(t)=t^{3}\Big[c_1\cos(4\ln t)+c_2\sin(4\ln t)\Big],\quad t>0$$