

# Series Solutions

We return now to the general homogeneous linear second-order equation

$$L[y] = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \quad (1)$$

with  $P(t)$  unequal to zero in the interval  $\alpha < t < \beta$ . It was shown in Section 2.1 that every solution  $y(t)$  of (1) can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

where  $y_1(t)$  and  $y_2(t)$  are any two linearly independent solutions of (1). Thus, the problem of finding all solutions of (1) is reduced to the simpler problem of finding just two solutions. In Section 2.2 we handled the special case where  $P$ ,  $Q$ , and  $R$  are constants. The next simplest case is when  $P(t)$ ,  $Q(t)$ , and  $R(t)$  are polynomials in  $t$ .

EXAMPLE: Find two linearly independent solutions of the equation

$$L[y] = \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} - 2y = 0 \quad (2)$$

Solution: We set

$$y(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \dots = \sum_{n=0}^{\infty} a_n t^n$$

Computing

$$\frac{dy}{dt} = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \dots = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

and

$$\frac{d^2y}{dt^2} = 2a_2 + 6a_3t + 12a_4t^2 + \dots = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}$$

we see that  $y(t)$  is a solution of (2) if

$$\begin{aligned} L[y](t) &= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2t \sum_{n=0}^{\infty} n a_n t^{n-1} - 2 \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2 \sum_{n=0}^{\infty} n a_n t^n - 2 \sum_{n=0}^{\infty} a_n t^n = 0 \end{aligned} \quad (3)$$

Our next step is to rewrite the first summation in (3) so that the exponent of the general term is  $n$ , instead of  $n-2$ . We have

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} &= \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} t^n \\ &= (-2+2)(-2+1)a_{-2+2} t^{-2} + (-1+2)(-1+1)a_{-1+2} t^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n \\ &= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n \end{aligned}$$

therefore we can rewrite (3) in the form

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n - 2 \sum_{n=0}^{\infty} na_n t^n - 2 \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n - 2a_n]t^n = 0 \quad (4)$$

Setting the sum of the coefficients of like powers of  $t$  in (4) equal to zero gives

$$(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0$$

so that

$$a_{n+2} = \frac{2na_n + 2a_n}{(n+2)(n+1)} = \frac{2(n+1)a_n}{(n+2)(n+1)} = \frac{2a_n}{n+2} \quad (5)$$

To find two solutions of (2), we choose two different sets of values of  $a_0$ , and  $a_1$ . The simplest possible choices are (i)  $a_0 = 1, a_1 = 0$ ; (ii)  $a_0 = 0, a_1 = 1$ .

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In this case, all the odd coefficients  $a_1, a_3, a_5, \dots$  are zero since

$$a_1 = 0, \quad a_3 = \frac{2a_1}{3} = \frac{2 \cdot 0}{3} = 0, \quad a_5 = \frac{2a_3}{5} = \frac{2 \cdot 0}{5} = 0$$

and so on. The even coefficients are determined from the relations

$$a_2 = \frac{2a_0}{2} = a_0 = 1, \quad a_4 = \frac{2a_2}{4} = \frac{a_2}{2} = \frac{1}{2}, \quad a_6 = \frac{2a_4}{6} = \frac{a_4}{3} = \frac{1}{2 \cdot 3}$$

and so on. Proceeding inductively, we find that

$$a_{2n} = \frac{1}{2 \cdot 3 \dots n} = \frac{1}{n!}$$

Hence,

$$y_1(t) = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \dots = e^{t^2}$$

is one solution of (2).

(ii)  $a_0 = 0, a_1 = 1$ .

In this case, all the even coefficients  $a_0, a_2, a_4, \dots$  are zero since

$$a_0 = 0, \quad a_2 = \frac{2a_0}{2} = \frac{2 \cdot 0}{2} = 0, \quad a_4 = \frac{2a_2}{4} = \frac{2 \cdot 0}{4} = 0$$

and so on. The odd coefficients are determined from the relations

$$a_1 = 1, \quad a_3 = \frac{2a_1}{3} = \frac{2}{3}, \quad a_5 = \frac{2a_3}{5} = \frac{2}{5} \cdot \frac{2}{3}, \quad a_7 = \frac{2a_5}{7} = \frac{2}{7} \cdot \frac{2}{5} \cdot \frac{2}{3}$$

and so on. Proceeding inductively, we find that

$$a_{2n+1} = \frac{2^n}{3 \cdot 5 \dots (2n+1)}$$

Hence,

$$y_2(t) = t + \frac{2t^3}{3} + \frac{2^2 t^5}{3 \cdot 5} + \dots = \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{3 \cdot 5 \dots (2n+1)}$$

is a second solution of (2).

**THEOREM 6:** Let the variable  $t$  assume complex values, and let  $z_0$ , be the point closest to  $t_0$  at which  $f$  or one of its derivatives fails to exist. Compute the distance  $\rho$ , in the complex plane, between  $t_0$  and  $z_0$ . Then, the Taylor series of  $f$  about  $t_0$  converges for  $|t - t_0| < \rho$ , and diverges for  $|t - t_0| > \rho$ .

**THEOREM 7:** Let the functions  $Q(t)/P(t)$  and  $R(t)/P(t)$  have convergent Taylor series expansions about  $t = t_0$ , for  $|t - t_0| < \rho$ . Then, every solution  $y(t)$  of the differential equation

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \quad (6)$$

is analytic at  $t = t_0$ , and the radius of convergence of its Taylor series expansion about  $t = t_0$  is at least  $\rho$ . The coefficients  $a_2, a_3, \dots$ , in the Taylor series expansion

$$y(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \dots \quad (7)$$

are determined by plugging the series (7) into the differential equation (6) and setting the sum of the coefficients of like powers of  $t$  in this expression equal to zero.

**EXAMPLE:**

(a) Find two linearly independent solutions of the equation

$$L[y] = \frac{d^2y}{dt^2} + \frac{3t}{1+t^2}\frac{dy}{dt} + \frac{1}{1+t^2}y = 0 \quad (8)$$

(b) Find the solution  $y(t)$  of (8) which satisfies the initial conditions  $y(0) = 2$ ,  $y'(0) = 3$ .

Solution: We first multiply both sides of (8) by  $1 + t^2$  to obtain the equivalent equation

$$L[y] = (1 + t^2)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y = 0 \quad (9)$$

and set

$$y(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \dots = \sum_{n=0}^{\infty} a_n t^n$$

We have (see the example above)

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} n a_n t^{n-1} \quad \text{and} \quad \frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

therefore  $y(t)$  is a solution of (9) if

$$\begin{aligned} L[y](t) &= (1 + t^2) \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + 3t \sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + t^2 \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + 3t \sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} t^2 + \sum_{n=0}^{\infty} 3n a_n t^{n-1} t + \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n t^n + \sum_{n=0}^{\infty} 3n a_n t^n + \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} [n(n-1) + 3n + 1] a_n t^n = 0 \end{aligned}$$

Note that

$$n(n-1) + 3n + 1 = n^2 - n + 3n + 1 = n^2 + 2n + 1 = (n+1)^2$$

therefore

$$L[y](t) = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} (n+1)^2 a_n t^n = 0 \quad (10)$$

In the previous example it was shown that

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

therefore we can rewrite (10) in the form

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} (n+1)^2 a_n t^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)^2 a_n] t^n = 0 \quad (11)$$

Setting the sum of the coefficients of like powers of  $t$  in (11) equal to zero gives

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_n = 0$$

so that

$$a_{n+2} = -\frac{(n+1)^2 a_n}{(n+2)(n+1)} = -\frac{(n+1)a_n}{n+2} \quad (12)$$

To find two solutions of (8), we choose two different sets of values of  $a_0$ , and  $a_1$ . The simplest possible choices are (i)  $a_0 = 1, a_1 = 0$ ; (ii)  $a_0 = 0, a_1 = 1$ .

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In this case, all the odd coefficients  $a_1, a_3, a_5, \dots$  are zero since

$$a_1 = 0, \quad a_3 = -\frac{2a_1}{3} = -\frac{2 \cdot 0}{3} = 0, \quad a_5 = -\frac{4a_3}{5} = -\frac{4 \cdot 0}{5} = 0$$

and so on. The even coefficients are determined from the relations

$$a_0 = 1, \quad a_2 = -\frac{a_0}{2} = -\frac{1}{2}, \quad a_4 = -\frac{3a_2}{4} = \frac{1 \cdot 3}{2 \cdot 4}, \quad a_6 = -\frac{5a_4}{6} = -\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}$$

and so on. Proceeding inductively, we find that

$$a_{2n} = (-1)^n \frac{1 \cdot 3 \dots (2n-1)}{2 \cdot 4 \dots 2n} = (-1)^n \frac{1 \cdot 3 \dots (2n-1)}{2^n n!}$$

Thus,

$$y_1(t) = 1 - \frac{t^2}{2} + \frac{1 \cdot 3}{2 \cdot 4} t^4 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \dots (2n-1)}{2^n n!} t^{2n} \quad (13)$$

is one solution of (8). The ratio of the  $(n+1)$ st term to the  $n$ th term of  $y_1(t)$  is

$$-\frac{1 \cdot 3 \dots (2n-1)(2n+1)t^{2n+2}}{2^{n+1}(n+1)!} \times \frac{2^n n!}{1 \cdot 3 \dots (2n-1)t^{2n}} = -\frac{(2n+1)t^2}{2(n+1)}$$

and the absolute value of this quantity approaches  $t^2$  as  $n$  approaches infinity. Hence, by the Cauchy ratio test, the infinite series (13) converges for  $|t| < 1$ , and diverges for  $|t| > 1$ .

(ii)  $a_0 = 0$ ,  $a_1 = 1$ .

In this case, all the even coefficients  $a_0, a_2, a_4, \dots$  are zero since

$$a_0 = 0, \quad a_2 = -\frac{a_0}{2} = -\frac{0}{2} = 0, \quad a_4 = -\frac{3a_2}{4} = -\frac{3 \cdot 0}{4} = 0$$

and so on. The odd coefficients are determined from the relations

$$a_1 = 1, \quad a_3 = -\frac{2a_1}{3} = -\frac{2}{3}, \quad a_5 = -\frac{4a_3}{5} = \frac{2 \cdot 4}{3 \cdot 5}, \quad a_7 = -\frac{6a_5}{7} = -\frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}$$

and so on. Proceeding inductively, we find that

$$a_{2n+1} = (-1)^n \frac{2 \cdot 4 \dots 2n}{3 \cdot 5 \dots (2n+1)} = (-1)^n \frac{2^n n!}{3 \cdot 5 \dots (2n+1)}$$

Thus,

$$y_2(t) = t - \frac{2}{3}t^3 + \frac{2 \cdot 4}{3 \cdot 5}t^5 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{2^n n!}{3 \cdot 5 \dots (2n+1)} t^{2n+1}$$

is one solution of (8). The ratio of the  $(n+1)$ st term to the  $n$ th term of  $y_2(t)$  is

$$-\frac{2^{n+1}(n+1)!t^{2n+3}}{3 \cdot 5 \dots (2n+1)(2n+3)} \times \frac{3 \cdot 5 \dots (2n+1)}{2^n n! t^{2n+1}} = -\frac{2(n+1)t^2}{2n+3}$$

and the absolute value of this quantity approaches  $t^2$  as  $n$  approaches infinity. Hence, by the Cauchy ratio test, the infinite series (13) converges for  $|t| < 1$ , and diverges for  $|t| > 1$ .

(b) The solution  $y_1(t)$  satisfies the initial conditions  $y_1(0) = 1$ ,  $y_1'(0) = 0$ , while  $y_2(t)$  satisfies the initial conditions  $y_2(0) = 0$ ,  $y_2'(0) = 1$ . Hence

$$y(t) = 2y_1(t) + 3y_2(t)$$

satisfies the initial conditions  $y(0) = 2$ ,  $y'(0) = 3$ .

# Singular points, Euler equations

The differential equation

$$L[y] = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \quad (1)$$

is said to be singular at  $t = t_0$  if  $P(t_0) = 0$ . Solutions of (1) frequently become very large, or oscillate very rapidly, in a neighborhood of the singular point  $t_0$ . Thus, solutions of (1) may not even be continuous, let alone analytic at  $t_0$ , and the method of power series solution will fail to work, in general.

Our goal is to find a class of singular equations which we can solve for  $t$  near  $t_0$ . To this end we will first study a very simple equation, known as Euler's equation, which is singular, but easily solvable.

DEFINITION: The differential equation

$$L[y] = t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0 \quad (2)$$

where  $\alpha$  and  $\beta$  are constants is known as Euler's equation.

We will assume, for simplicity, that  $t > 0$ .

**Case I:**  $(\alpha - 1)^2 - 4\beta > 0$ . In this case the general solution of (2) is

$$y(t) = c_1t^{r_1} + c_2t^{r_2}$$

where  $r_1$  and  $r_2$  are roots of the equation

$$r^2 + (\alpha - 1)r + \beta = 0$$

EXAMPLE: Find the general solution of

$$L[y] = t^2\frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 0, \quad t > 0$$

Solution: Here  $\alpha = 4$  and  $\beta = 2$ . One can check that  $(\alpha - 1)^2 - 4\beta > 0$ , therefore the general solution is

$$y(t) = c_1t^{-1} + c_2t^{-2}$$

since  $-1$  and  $-2$  are roots of the equation

$$r^2 + 3r + 2 = 0$$

**Case II:**  $(\alpha - 1)^2 - 4\beta = 0$ . In this case the general solution of (2) is

$$y(t) = (c_1 + c_2 \ln t)t^r, \quad t > 0$$

where

$$r = \frac{1 - \alpha}{2}$$

EXAMPLE: Find the general solution of

$$L[y] = t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0, \quad t > 0$$

Solution: Here  $\alpha = -5$  and  $\beta = 9$ . One can check that  $(\alpha - 1)^2 - 4\beta = 0$ , therefore the general solution is

$$y(t) = (c_1 + c_2 \ln t)t^{(1+5)/2} = (c_1 + c_2 \ln t)t^3, \quad t > 0$$

**Case III:**  $(\alpha - 1)^2 - 4\beta < 0$ . In this case the general solution is

$$y(t) = t^\lambda \left[ c_1 \cos(\mu \ln t) + c_2 \sin(\mu \ln t) \right]$$

where

$$\lambda = \frac{1 - \alpha}{2}, \quad \mu = \frac{[4\beta - (\alpha - 1)^2]^{1/2}}{2}$$

EXAMPLE: Find the general solution of

$$L[y] = t^2 \frac{d^2 y}{dt^2} - 5t \frac{dy}{dt} + 25y = 0, \quad t > 0$$

Solution: Here  $\alpha = -5$  and  $\beta = 25$ . One can check that  $(\alpha - 1)^2 - 4\beta < 0$ . We have

$$\lambda = \frac{1 + 5}{2} = 3, \quad \mu = \frac{[4 \cdot 25 - (-5 - 1)^2]^{1/2}}{2} = 4$$

therefore the general solution is

$$y(t) = t^3 \left[ c_1 \cos(4 \ln t) + c_2 \sin(4 \ln t) \right], \quad t > 0$$