The Method of Variation of Parameters

Consider the second-order linear nonhomogeneous differential equation

$$y'' + a_1 y' + a_2 y = F \tag{1}$$

where we assume that a_1, a_2, and F are continuous on an interval I. Suppose that $y = y_1(x)$ and $y = y_2(x)$ are two linearly independent solutions to the associated homogeneous equation

$$y'' + a_1 y' + a_2 y = 0 \tag{2}$$

on I, so that the general solution to equation (2) on I is

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

The variation-of-parameters method consists of replacing the constants c_1 and c_2 by functions $u_1(x)$ and $u_2(x)$ (that is, we allow the parameters c_1 and c_2 to vary), determined in such a way that the resulting function

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{3}$$

is a particular solution to equation (1).

Differentiating equation (3) with respect to x yields

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

It is tempting to differentiate this expression once more and then substitute into equation (1) to determine u_1 and u_2. However, if we did this, the resulting expression for y_p'' would involve second derivatives of u_1 and u_2, hence we would have complicated our problem. Since y_p contains two unknown functions, whereas equation (1) gives only one condition for determining them, we have the freedom to impose a further constraint on u_1 and u_2. In order to eliminate second derivatives of u_1 and u_2 arising in y_p we try for solutions of the form (3) satisfying the constraint

$$u_1' y_1 + u_2' y_2 = 0 \tag{4}$$

The expression for y_p' then reduces to

$$y_p' = u_1 y_1' + u_2 y_2'$$

so that

$$y_p'' = u_1' y' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Substituting into equation (1) and collecting terms yields

$$u_1(y_1'' + a_1 y_1' + a_2 y_1) + u_2(y_2'' + a_1 y_2' + a_2 y_2) + (u_1' y_1' + u_2' y_2') = F(x)$$

The terms multiplying u_1 and u_2 vanish, since y_1 and y_2 each solve $y'' + a_1 y' + a_2 y = 0$. We therefore require that

$$u_1' y_1' + u_2' y_2' = F \tag{5}$$

Consequently, $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ is a solution to equation (1), provided that u_1 and u_2 satisfy equations (4) and (5). That is,

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = F \end{cases} \tag{6}$$

This is a linear system of equations for the unknowns u_1' and u_2'. The matrix of coefficients of this system has determinant

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

which is the Wronskian, $W[y_1, y_2](x)$, of y_1 and y_2. Since y_1 and y_2 are linearly independent on I, $W[y_1, y_2](x)$ is *nonzero* on I and hence the system (6) has a unique solution for u_1' and u_2'. Indeed, applying Cramer's rule to (6) yields

$$u_1'(x) = -\frac{y_2(x)F(x)}{W[y_1, y_2](x)}, \qquad u_2'(x) = \frac{y_1(x)F(x)}{W[y_1, y_2](x)} \tag{7}$$

Finally, we obtain $u_1(t)$ and $u_2(t)$ by integrating the right-hand sides of (7).

EXAMPLE: Solve $y'' + y = \sec x, \ -\pi/2 < x < \pi/2$.

Solution: The characteristic polynomial is

$$P(r) = r^2 + 1$$

We have

$$r^2 + 1 = 0 \qquad \Longrightarrow \qquad r^2 = -1 \qquad \Longrightarrow \qquad r = \pm\sqrt{-1} = 0 \pm 1 \cdot i$$

So, $\alpha = 0$ and $\beta = 1$. Consequently, two linearly independent solutions to the associated homogeneous equation are

$$y_1(x) = e^{\alpha x}\cos\beta x = e^{0\cdot x}\cos(1\cdot x) = \cos x \qquad \text{and} \qquad y_2(x) = e^{\alpha x}\sin\beta x = e^{0\cdot x}\sin(1\cdot x) = \sin x$$

Thus, a particular solution to the given differential equation is

$$y_p(x) = u_1 y_1 + u_2 y_2 = u_1 \cos x + u_2 \sin x \tag{8}$$

where u_1 and u_2 satisfy

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = F \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \cos x\, u_1' + \sin x\, u_2' &= 0 \\ -\sin x\, u_1' + \cos x\, u_2' &= \sec x \end{cases}$$

Applying Cramer's rule (or formulas (7)), the solution to this system is

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0 - \sin x \sec x}{\cos^2 x + \sin^2 x} = -\sin x \sec x = -\frac{\sin x}{\cos x}$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec x - 0}{\cos^2 x + \sin^2 x} = \cos x \sec x = \cos x\,\frac{1}{\cos x} = 1$$

Consequently,

$$u_1(x) = -\int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} \cdot (-\sin x) dx = \left[\begin{array}{c} \cos x = u \\ d\cos x = du \\ -\sin x dx = du \end{array} \right] = \int \frac{1}{u} du = \ln |u| = \ln |\cos x|$$

and

$$u_2(x) = \int 1 dx = x$$

where we have set the integration constants to zero, since we require only one particular solution. Substitution into equation (8) yields

$$\begin{aligned} y_p(x) &= u_1 \cos x + u_2 \sin x \\ &= \ln |\cos x| \cdot \cos x + x \sin x \\ &= \ln(\cos x) \cdot \cos x + x \sin x \qquad\qquad (-\pi/2 < x < \pi/2) \end{aligned}$$

so that the general solution to the given differential equation is

$$y(x) = c_1 \cos x + c_2 \sin x + \ln(\cos x) \cdot \cos x + x \sin x$$

EXAMPLE: Solve $y'' + 4y' + 4y = e^{-2x} \ln x, \ x > 0$.

Solution: The characteristic polynomial is

$$P(r) = r^2 + 4r + 4 = r^2 + 2r \cdot 2 + 2^2 = (r+2)^2$$

Thus, $r = -2$ is a repeated root of the characteristic equation, and therefore two linearly independent solutions to the associated homogeneous equation are

$$y_1(x) = e^{-2x} \qquad \text{and} \qquad y_2(x) = xe^{-2x}$$

hence we seek a particular solution to the given differential equation of the form

$$y_p(x) = u_1 y_1 + u_2 y_2 = u_1 e^{-2x} + u_2 x e^{-2x}$$

where u_1 and u_2 satisfy

$$\left\{ \begin{array}{l} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = F \end{array} \right. \qquad \Longrightarrow \qquad \left\{ \begin{array}{r} e^{-2x} u_1' + x e^{-2x} u_2' = 0 \\ -2e^{-2x} u_1' + e^{-2x}(1-2x) u_2' = e^{-2x} \ln x \end{array} \right.$$

WORK: $(xe^{-2x})' = x'e^{-2x} + x(e^{-2x})' = 1 \cdot e^{-2x} + x(-2)e^{-2x} = e^{-2x}(1-2x)$

Canceling out e^{-2x}, we get

$$\left\{ \begin{array}{r} u_1' + x u_2' = 0 \\ -2u_1' + (1-2x) u_2' = \ln x \end{array} \right.$$

Applying Cramer's rule (or formulas (7)), the solution to this system is

$$u_1' = \frac{\begin{vmatrix} 0 & x \\ \ln x & 1-2x \end{vmatrix}}{\begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix}} = \frac{0 - x\ln x}{1\cdot(1-2x) - x\cdot(-2)} = \frac{-x\ln x}{1-2x+2x} = -x\ln x$$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ -2 & \ln x \end{vmatrix}}{\begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix}} = \frac{1\cdot\ln x - 0}{1\cdot(1-2x) - x\cdot(-2)} = \frac{\ln x}{1-2x+2x} = \ln x$$

So,

$$u_1' = -x\ln x, \qquad u_2' = \ln x$$

Integrating both of these expressions by parts

$$\int u dv = uv - \int v du$$

we obtain

$$\begin{aligned} u_1(x) = -\int x\ln x dx = \left[\begin{array}{c|c} \ln x = u & x dx = dv \\ d(\ln x) = du & \dfrac{x^2}{2} = v \\ \dfrac{1}{x}dx = du & \end{array}\right] &= -\ln x\cdot\frac{x^2}{2} + \int \frac{x^2}{2}\cdot\frac{1}{x}dx \\ &= -\frac{1}{2}x^2\ln x + \frac{1}{2}\int x dx \\ &= -\frac{1}{2}x^2\ln x + \frac{1}{2}\cdot\frac{x^2}{2} \\ &= -\frac{1}{2}x^2\ln x + \frac{1}{4}x^2 \\ &= \frac{1}{4}x^2\cdot 1 - \frac{1}{4}x^2\cdot 2\ln x \\ &= \frac{1}{4}x^2(1-2\ln x) \end{aligned}$$

and

$$\begin{aligned} u_2(x) = \int \ln x dx = \left[\begin{array}{c|c} \ln x = u & dx = dv \\ d(\ln x) = du & x = v \\ \dfrac{1}{x}dx = du & \end{array}\right] = \ln x\cdot x - \int x\cdot\frac{1}{x}dx &= x\ln x - \int dx \\ &= x\ln x - x \\ &= x(\ln x - 1) \end{aligned}$$

So,

$$u_1(x) = \frac{1}{4}x^2(1 - 2\ln x), \qquad u_2(x) = x(\ln x - 1)$$

Thus,

$$\begin{aligned}
y_p(x) &= u_1 e^{-2x} + u_2 x e^{-2x} \\
&= \frac{1}{4}x^2(1 - 2\ln x) \cdot e^{-2x} + x(\ln x - 1) \cdot x e^{-2x} \\
&= \frac{1}{4}x^2 e^{-2x}(1 - 2\ln x) + x^2 e^{-2x}(\ln x - 1) \\
&= \frac{1}{4}x^2 e^{-2x}(1 - 2\ln x) + \frac{1}{4}x^2 e^{-2x}(4\ln x - 4) \\
&= \frac{1}{4}x^2 e^{-2x}(1 - 2\ln x + 4\ln x - 4) \\
&= \frac{1}{4}x^2 e^{-2x}(2\ln x - 3)
\end{aligned}$$

Consequently, the general solution to the given differential equation is

$$\begin{aligned}
y(x) &= c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{4}x^2 e^{-2x}(2\ln x - 3) \\
&= e^{-2x}\left[c_1 + c_2 x + \frac{1}{4}x^2(2\ln x - 3)\right]
\end{aligned}$$