

# The Method of Variation of Parameters

Consider the second-order linear nonhomogeneous differential equation

$$y'' + a_1y' + a_2y = F \tag{1}$$

where we assume that  $a_1, a_2$ , and  $F$  are continuous on an interval  $I$ . Suppose that  $y = y_1(x)$  and  $y = y_2(x)$  are two linearly independent solutions to the associated homogeneous equation

$$y'' + a_1y' + a_2y = 0 \tag{2}$$

on  $I$ , so that the general solution to equation (2) on  $I$  is

$$y_c = c_1y_1(x) + c_2y_2(x)$$

The variation-of-parameters method consists of replacing the constants  $c_1$  and  $c_2$  by functions  $u_1(x)$  and  $u_2(x)$  (that is, we allow the parameters  $c_1$  and  $c_2$  to vary), determined in such a way that the resulting function

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{3}$$

is a particular solution to equation (1).

Differentiating equation (3) with respect to  $x$  yields

$$y'_p = u'_1y_1 + u_1y'_1 + u'_2y_2 + u_2y'_2$$

It is tempting to differentiate this expression once more and then substitute into equation (1) to determine  $u_1$  and  $u_2$ . However, if we did this, the resulting expression for  $y''_p$  would involve second derivatives of  $u_1$  and  $u_2$ , hence we would have complicated our problem. Since  $y_p$  contains two unknown functions, whereas equation (1) gives only one condition for determining them, we have the freedom to impose a further constraint on  $u_1$  and  $u_2$ . In order to eliminate second derivatives of  $u_1$  and  $u_2$  arising in  $y_p$  we try for solutions of the form (3) satisfying the constraint

$$u'_1y_1 + u'_2y_2 = 0 \tag{4}$$

The expression for  $y'_p$  then reduces to

$$y'_p = u_1y'_1 + u_2y'_2$$

so that

$$y''_p = u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2$$

Substituting into equation (1) and collecting terms yields

$$u_1(y''_1 + a_1y'_1 + a_2y_1) + u_2(y''_2 + a_1y'_2 + a_2y_2) + (u'_1y'_1 + u'_2y'_2) = F(x)$$

The terms multiplying  $u_1$  and  $u_2$  vanish, since  $y_1$  and  $y_2$  each solve  $y'' + a_1y' + a_2y = 0$ . We therefore require that

$$u'_1y'_1 + u'_2y'_2 = F \tag{5}$$

Consequently,  $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$  is a solution to equation (1), provided that  $u_1$  and  $u_2$  satisfy equations (4) and (5). That is,

$$\begin{cases} y_1u'_1 + y_2u'_2 = 0 \\ y'_1u'_1 + y'_2u'_2 = F \end{cases} \tag{6}$$

This is a linear system of equations for the unknowns  $u'_1$  and  $u'_2$ . The matrix of coefficients of this system has determinant

$$\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

which is the Wronskian,  $W[y_1, y_2](x)$ , of  $y_1$  and  $y_2$ . Since  $y_1$  and  $y_2$  are linearly independent on  $I$ ,  $W[y_1, y_2](x)$  is *nonzero* on  $I$  and hence the system (6) has a unique solution for  $u'_1$  and  $u'_2$ . Indeed, applying Cramer's rule to (6) yields

$$u'_1(x) = -\frac{y_2(x)F(x)}{W[y_1, y_2](x)}, \quad u'_2(x) = \frac{y_1(x)F(x)}{W[y_1, y_2](x)} \quad (7)$$

Finally, we obtain  $u_1(t)$  and  $u_2(t)$  by integrating the right-hand sides of (7).

EXAMPLE: Solve  $y'' + y = \sec x$ ,  $-\pi/2 < x < \pi/2$ .

Solution: The characteristic polynomial is

$$P(r) = r^2 + 1$$

We have

$$r^2 + 1 = 0 \quad \implies \quad r^2 = -1 \quad \implies \quad r = \pm\sqrt{-1} = 0 \pm 1 \cdot i$$

So,  $\alpha = 0$  and  $\beta = 1$ . Consequently, two linearly independent solutions to the associated homogeneous equation are

$$y_1(x) = e^{\alpha x} \cos \beta x = e^{0 \cdot x} \cos(1 \cdot x) = \cos x \quad \text{and} \quad y_2(x) = e^{\alpha x} \sin \beta x = e^{0 \cdot x} \sin(1 \cdot x) = \sin x$$

Thus, a particular solution to the given differential equation is

$$y_p(x) = u_1 y_1 + u_2 y_2 = u_1 \cos x + u_2 \sin x \quad (8)$$

where  $u_1$  and  $u_2$  satisfy

$$\begin{cases} y_1 u'_1 + y_2 u'_2 = 0 \\ y'_1 u'_1 + y'_2 u'_2 = F \end{cases} \quad \implies \quad \begin{cases} \cos x u'_1 + \sin x u'_2 = 0 \\ -\sin x u'_1 + \cos x u'_2 = \sec x \end{cases}$$

Applying Cramer's rule (or formulas (7)), the solution to this system is

$$u'_1 = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0 - \sin x \sec x}{\cos^2 x + \sin^2 x} = -\sin x \sec x = -\frac{\sin x}{\cos x}$$

$$u'_2 = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec x - 0}{\cos^2 x + \sin^2 x} = \cos x \sec x = \cos x \frac{1}{\cos x} = 1$$

Consequently,

$$u_1(x) = - \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} \cdot (-\sin x) dx = \begin{bmatrix} \cos x = u \\ d \cos x = du \\ -\sin x dx = du \end{bmatrix} = \int \frac{1}{u} du = \ln |u| = \ln |\cos x|$$

and

$$u_2(x) = \int 1 dx = x$$

where we have set the integration constants to zero, since we require only one particular solution. Substitution into equation (8) yields

$$\begin{aligned} y_p(x) &= u_1 \cos x + u_2 \sin x = \ln |\cos x| \cdot \cos x + x \sin x \\ &= \ln(\cos x) \cdot \cos x + x \sin x \quad (-\pi/2 < x < \pi/2) \end{aligned}$$

so that the general solution to the given differential equation is

$$y(x) = c_1 \cos x + c_2 \sin x + \ln(\cos x) \cdot \cos x + x \sin x$$

EXAMPLE: Solve  $y'' + 4y' + 4y = e^{-2x} \ln x$ ,  $x > 0$ .

Solution: The characteristic polynomial is

$$P(r) = r^2 + 4r + 4 = r^2 + 2r \cdot 2 + 2^2 = (r + 2)^2$$

Thus,  $r = -2$  is a repeated root of the characteristic equation, and therefore two linearly independent solutions to the associated homogeneous equation are

$$y_1(x) = e^{-2x} \quad \text{and} \quad y_2(x) = xe^{-2x}$$

hence we seek a particular solution to the given differential equation of the form

$$y_p(x) = u_1 y_1 + u_2 y_2 = u_1 e^{-2x} + u_2 x e^{-2x}$$

where  $u_1$  and  $u_2$  satisfy

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = F \end{cases} \implies \begin{cases} e^{-2x} u_1' + x e^{-2x} u_2' = 0 \\ -2e^{-2x} u_1' + e^{-2x} (1 - 2x) u_2' = e^{-2x} \ln x \end{cases}$$

WORK:  $(xe^{-2x})' = x'e^{-2x} + x(e^{-2x})' = 1 \cdot e^{-2x} + x(-2)e^{-2x} = e^{-2x}(1 - 2x)$

Applying Cramer's rule (or formulas (7)), the solution to this system is

$$\begin{aligned} u_1' &= \frac{\begin{vmatrix} 0 & x e^{-2x} \\ e^{-2x} \ln x & e^{-2x} (1 - 2x) \end{vmatrix}}{\begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & e^{-2x} (1 - 2x) \end{vmatrix}} = \frac{0 - x e^{-2x} \cdot e^{-2x} \ln x}{e^{-2x} \cdot e^{-2x} (1 - 2x) + x e^{-2x} \cdot 2e^{-2x}} = \frac{-x e^{-4x} \ln x}{e^{-4x}} = -x \ln x \\ u_2' &= \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x} \ln x \end{vmatrix}}{\begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & e^{-2x} (1 - 2x) \end{vmatrix}} = \frac{e^{-2x} \cdot e^{-2x} \ln x - 0}{e^{-2x} \cdot e^{-2x} (1 - 2x) + x e^{-2x} \cdot 2e^{-2x}} = \frac{e^{-4x} \ln x}{e^{-4x}} = \ln x \end{aligned}$$

WORK:

$$e^{-2x} \cdot e^{-2x}(1 - 2x) + xe^{-2x} \cdot 2e^{-2x} = e^{-4x}(1 - 2x) + 2xe^{-4x} = e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}$$

So,

$$u'_1 = -x \ln x, \quad u'_2 = \ln x$$

Integrating both of these expressions by parts

$$\int u dv = uv - \int v du$$

we obtain

$$\begin{aligned} u_1(x) = - \int x \ln x dx &= \left[ \begin{array}{l|l} \ln x = u & x dx = dv \\ d(\ln x) = du & \frac{x^2}{2} = v \\ \frac{1}{x} dx = du & \end{array} \right] = - \ln x \cdot \frac{x^2}{2} + \int \frac{x^2}{2} \cdot \frac{1}{x} dx = -\frac{1}{2}x^2 \ln x + \frac{1}{2} \int x dx \\ &= -\frac{1}{2}x^2 \ln x + \frac{1}{2} \cdot \frac{x^2}{2} = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 = \frac{1}{4}x^2 \cdot 1 - \frac{1}{4}x^2 \cdot 2 \ln x = \frac{1}{4}x^2(1 - 2 \ln x) \end{aligned}$$

and

$$\begin{aligned} u_2(x) = \int \ln x dx &= \left[ \begin{array}{l|l} \ln x = u & dx = dv \\ d(\ln x) = du & x = v \\ \frac{1}{x} dx = du & \end{array} \right] = \ln x \cdot x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx \\ &= x \ln x - x = x(\ln x - 1) \end{aligned}$$

So,

$$u_1(x) = \frac{1}{4}x^2(1 - 2 \ln x), \quad u_2(x) = x(\ln x - 1)$$

Thus,

$$\begin{aligned} y_p(x) &= u_1 e^{-2x} + u_2 x e^{-2x} \\ &= \frac{1}{4}x^2(1 - 2 \ln x) \cdot e^{-2x} + x(\ln x - 1) \cdot x e^{-2x} \\ &= \frac{1}{4}x^2 e^{-2x}(1 - 2 \ln x) + x^2 e^{-2x}(\ln x - 1) \\ &= \frac{1}{4}x^2 e^{-2x}(1 - 2 \ln x) + \frac{1}{4}x^2 e^{-2x}(4 \ln x - 4) \\ &= \frac{1}{4}x^2 e^{-2x}(1 - 2 \ln x + 4 \ln x - 4) \\ &= \frac{1}{4}x^2 e^{-2x}(2 \ln x - 3) \end{aligned}$$

Consequently, the general solution to the given differential equation is

$$\begin{aligned} y(x) &= c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{4}x^2 e^{-2x}(2 \ln x - 3) \\ &= e^{-2x} \left[ c_1 + c_2 x + \frac{1}{4}x^2(2 \ln x - 3) \right] \end{aligned}$$