The Nonhomogeneous Equation

We turn our attention now to the nonhomogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{1}$$

where the functions $p(t)$, $q(t)$ and $g(t)$ are continuous on an open interval $\alpha < t < \beta$.

THEOREM: Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \tag{2}$$

and let $\psi(t)$ be any particular solution of the nonhomogeneous equation (1). Then, every solution $y(t)$ of (1) must be of the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t)$$

for some choice of constants c_1, c_2.

The proof of the Theorem above relies heavily on the following lemma.

LEMMA: The difference of any two solutions of the nonhomogeneous equation (1) is a solution of the homogeneous equation (2).

Proof: Let $y_1(t)$ and $y_2(t)$ be two solutions of (1). By the linearity of L,

$$L[y_1 - y_2](t) = L[y_1](t) - L[y_2](t) = g(t) - g(t) = 0$$

Hence, $y_1(t) - y_2(t)$ is a solution of the homogeneous equation (2). ■

EXAMPLE: Three solutions of a certain second-order nonhomogeneous linear equation are

$$\psi_1(t) = t, \quad \psi_2(t) = t + e^t, \quad \text{and} \quad \psi_3(t) = 1 + t + e^t$$

Find the general solution of this equation.

Solution: By the Lemma above, the functions

$$\psi_2(t) - \psi_1(t) = e^t \quad \text{and} \quad \psi_3(t) - \psi_2(t) = 1$$

are solutions of the corresponding homogeneous equation. Moreover, these functions are obviously linearly independent. Therefore, by the Theorem above, every solution $y(t)$ of this equation must be of the form

$$y(t) = c_1e^t + c_2 + t$$