

# The Nonhomogeneous Equation

We turn our attention now to the nonhomogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \quad (1)$$

where the functions  $p(t)$ ,  $q(t)$  and  $g(t)$  are continuous on an open interval  $\alpha < t < \beta$ .

THEOREM: Let  $y_1(t)$  and  $y_2(t)$  be two linearly independent solutions of the homogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \quad (2)$$

and let  $\psi(t)$  be any particular solution of the nonhomogeneous equation (1). Then, every solution  $y(t)$  of (1) must be of the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t)$$

for some choice of constants  $c_1, c_2$ .

The proof of Theorem 5 relies heavily on the following lemma.

LEMMA: The difference of any two solutions of the nonhomogeneous equation (1) is a solution of the homogeneous equation (2).

Proof: Let  $y_1(t)$  and  $y_2(t)$  be two solutions of (1). By the linearity of  $L$ ,

$$L[y_1 - y_2](t) = L[y_1](t) - L[y_2](t) = g(t) - g(t) = 0$$

Hence,  $y_1(t) - y_2(t)$  is a solution of the homogeneous equation (2). ■

EXAMPLE: Three solutions of a certain second-order nonhomogeneous linear equation are

$$\psi_1(t) = t, \quad \psi_2(t) = t + e^t, \quad \text{and} \quad \psi_3(t) = 1 + t + e^t$$

Find the general solution of this equation.

Solution: By the Lemma above, the functions

$$\psi_2(t) - \psi_1(t) = e^t \quad \text{and} \quad \psi_3(t) - \psi_2(t) = 1$$

are solutions of the corresponding homogeneous equation. Moreover, these functions are obviously linearly independent. Therefore, by the Theorem above, every solution  $y(t)$  of this equation must be of the form

$$y(t) = c_1e^t + c_2 + t$$