

Linear Equations with Constant Coefficients

EXAMPLE: Determine all solutions to the differential equation

$$y'' + y' - 6y = 0$$

of the form $y(x) = e^{rx}$, where r is a constant. Use your solutions to determine the general solution to the differential equation.

Solution: We have

$$y(x) = e^{rx}$$

$$y'(x) = (e^{rx})' = e^{rx}(rx)' = re^{rx}$$

$$y''(x) = (y'(x))' = (re^{rx})' = r(e^{rx})' = r(re^{rx}) = r^2e^{rx}$$

therefore

$$y'' + y' - 6y = r^2e^{rx} + re^{rx} - 6e^{rx} = e^{rx}(r^2 + r - 6) = 0$$

Since $e^{rx} \neq 0$, it follows that $r^2 + r - 6 = 0$ or equivalently,

$$(r + 3)(r - 2) = 0$$

Hence, two solutions to the differential equation are

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-3x}$$

Furthermore, the Wronskian of these solutions is

$$\begin{aligned} W[y_1, y_2](x) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-3x} \\ (e^{2x})' & (e^{-3x})' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & (-3)e^{-3x} \end{vmatrix} = e^{2x} \cdot (-3)e^{-3x} - e^{-3x} \cdot 2e^{2x} \\ &= -3e^{2x-3x} - 2e^{-3x+2x} \\ &= -3e^{-x} - 2e^{-x} \\ &= -5e^{-x} \neq 0 \end{aligned}$$

so that they are linearly independent on any interval. It follows directly from Theorem 2 (Section 2.1) that the general solution to the differential equation is

$$y(x) = c_1e^{2x} + c_2e^{-3x}$$

Consider the homogeneous linear second-order equation with constant coefficients

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{1}$$

where a , b , and c are constants, with $a \neq 0$. Consider also the corresponding *characteristic equation* of (1)

$$P(r) = ar^2 + br + c = 0$$

Let r_1, r_2 be the roots of the *auxiliary equation*, so that

$$P(r) = (r - r_1)(r - r_2)$$

1. If r_1, r_2 are *real* and *distinct*, then the functions e^{r_1x}, e^{r_2x} are linearly independent solutions to (1) on any interval.

2. If r_1, r_2 are *real* and $r_1 = r_2 = r$, then the functions e^{rx}, xe^{rx} are linearly independent solutions to (1) on any interval.

3. If r_1 is *complex*, say $r_1 = \alpha + \beta i$ (α and β are real, with $\beta \neq 0$), then $r_2 = \alpha - \beta i$ and the functions

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x$$

are linearly independent solutions to (1) on any interval.

4. The general solution to (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are linearly independent solutions to (1).

EXAMPLE: Determine the general solution to $y'' - y' - 2y = 0$.

Solution: The auxiliary polynomial is

$$P(r) = r^2 - r - 2 = (r - 2)(r + 1)$$

Therefore, the auxiliary equation has roots $r_1 = 2, r_2 = -1$, so that two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-x}$$

Hence, the general solution to the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}$$

EXAMPLE: Determine the general solution to $y'' + 6y' + 25y = 0$.

Solution: The auxiliary equation is

$$r^2 + 6r + 25 = 0$$

with roots

$$\begin{aligned} r_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(1)(25)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 100}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm \sqrt{(64)(-1)}}{2} = \frac{-6 \pm \sqrt{64}\sqrt{-1}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i \end{aligned}$$

Consequently, two linearly independent real-valued solutions to the differential equation are

$$y_1(x) = e^{-3x} \cos 4x \quad \text{and} \quad y_2(x) = e^{-3x} \sin 4x$$

and the general solution to the differential equation is

$$y(x) = c_1 e^{-3x} \cos 4x + c_2 e^{-3x} \sin 4x = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x)$$

EXAMPLE: Solve the initial-value problem

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 4$$

EXAMPLE: Solve the initial-value problem

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 4$$

Solution: The auxiliary polynomial is

$$P(r) = r^2 + 4r + 4 = r^2 + 2r \cdot 2 + 2^2 = (r + 2)^2$$

Thus, $r = -2$ is a repeated root of the auxiliary equation, and therefore two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{-2x}, \quad y_2(x) = xe^{-2x}$$

Consequently, the general solution is

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} = e^{-2x}(c_1 + c_2x)$$

The initial condition $y(0) = 1$ implies that

$$y(0) = e^{-2 \cdot 0}(c_1 + c_2 \cdot 0) \implies 1 = 1 \cdot (c_1 + 0) \implies c_1 = 1$$

Thus,

$$y(x) = e^{-2x}(1 + c_2x)$$

Differentiating this expression yields

$$\begin{aligned} y'(x) &= \left(e^{-2x}(1 + c_2x) \right)' \\ &= (e^{-2x})'(1 + c_2x) + e^{-2x}(1 + c_2x)' \\ &= e^{-2x}(-2x)'(1 + c_2x) + e^{-2x}(1' + (c_2x)') \\ &= e^{-2x}(-2)(1 + c_2x) + e^{-2x}(0 + c_2) \\ &= -2e^{-2x}(1 + c_2x) + c_2e^{-2x} \end{aligned}$$

therefore the second initial condition implies

$$y'(0) = -2e^{-2 \cdot 0}(1 + c_2 \cdot 0) + c_2e^{-2 \cdot 0}$$

$$4 = -2(1 + 0) + c_2 \cdot 1$$

$$4 = -2 + c_2$$

so $c_2 = 6$. Hence, the unique solution to the given initial-value problem is

$$y(x) = e^{-2x}(1 + 6x)$$