Linear Equations with Constant Coefficients

EXAMPLE: Determine all solutions to the differential equation

$$y'' + y' - 6y = 0$$

of the form $y(x) = e^{rx}$, where r is a constant. Use your solutions to determine the general solution to the differential equation.

Solution: We have

$$y(x) = e^{rx}$$

$$y'(x) = (e^{rx})' = e^{rx}(rx)' = re^{rx}$$

$$y''(x) = (y'(x))' = (re^{rx})' = r(e^{rx})' = r(re^{rx}) = r^2e^{rx}$$

therefore

$$y'' + y' - 6y = r^2e^{rx} + re^{rx} - 6e^{rx} = e^{rx}(r^2 + r - 6) = 0$$

Since $e^{rx} \neq 0$, it follows that $r^2 + r - 6 = 0$ or equivalently,

$$(r + 3)(r - 2) = 0$$

Hence, two solutions to the differential equation are

$$y_1(x) = e^{2x} \qquad \text{and} \qquad y_2(x) = e^{-3x}$$

Since y_1 and y_2 are linearly independent (they are not constant constant multiples of each other), they form a fundamental set of solutions of $y'' + y' - 6y = 0$ by the Corollary to Theorem 3 from Section 2.1. That is,

$$y(x) = c_1e^{2x} + c_2e^{-3x}$$

is the general solution to the differential equation. Another way to show that y_1 and y_2 are linearly independent is to note that the Wronskian of y_1 and y_2 is

$$\begin{aligned} W[y_1, y_2](x) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-3x} \\ (e^{2x})' & (e^{-3x})' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & (-3)e^{-3x} \end{vmatrix} &= e^{2x} \cdot (-3)e^{-3x} - e^{-3x} \cdot 2e^{2x} \\ &= -3e^{2x-3x} - 2e^{-3x+2x} \\ &= -3e^{-x} - 2e^{-x} \\ &= -5e^{-x} \neq 0 \end{aligned}$$

so y_1 and y_2 are linearly independent by Theorem 3 from Section 2.1.

Consider the homogeneous linear second-order equation with constant coefficients

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{1}$$

where a, b, and c are constants, with $a \neq 0$. Consider also the corresponding *characteristic equation* of (1)

$$P(r) = ar^2 + br + c = 0$$

Let r_1, r_2 be the roots of the *characteristic polynomial* $P(r)$, so that

$$P(r) = (r - r_1)(r - r_2)$$

1. If r_1, r_2 are *real* and *distinct*, then the functions $e^{r_1 x}, e^{r_2 x}$ are linearly independent solutions to (1) on any interval.

2. If r_1, r_2 are *real* and $r_1 = r_2 = r$, then the functions e^{rx}, xe^{rx} are linearly independent solutions to (1) on any interval.

3. If r_1 is *complex*, say $r_1 = \alpha + \beta i$ (α and β are real, with $\beta \neq 0$), then $r_2 = \alpha - \beta i$ and the functions

$$e^{\alpha x} \cos \beta x, \qquad e^{\alpha x} \sin \beta x$$

are linearly independent solutions to (1) on any interval.

4. The general solution to (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are linearly independent solutions to (1).

EXAMPLE: Determine the general solution to $y'' - y' - 2y = 0$.

Solution: The characteristic polynomial is

$$P(r) = r^2 - r - 2 = (r-2)(r+1)$$

Therefore, the characteristic equation has roots $r_1 = 2, r_2 = -1$, so that two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{2x} \qquad \text{and} \qquad y_2(x) = e^{-x}$$

Hence, the general solution to the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}$$

EXAMPLE: Determine the general solution to $y'' + 6y' + 25y = 0$.

Solution: The characteristic equation is

$$r^2 + 6r + 25 = 0$$

with the roots

$$\begin{aligned} r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-6 \pm \sqrt{6^2 - 4(1)(25)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 100}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= \frac{-6 \pm \sqrt{(64)(-1)}}{2} = \frac{-6 \pm \sqrt{64}\sqrt{-1}}{2} = \frac{-6 \pm 8i}{2} = \frac{-6}{2} \pm \frac{8i}{2} = -3 \pm 4i \end{aligned}$$

Consequently, two linearly independent real-valued solutions to the differential equation are

$$y_1(x) = e^{-3x} \cos 4x \qquad \text{and} \qquad y_2(x) = e^{-3x} \sin 4x$$

and the general solution to the differential equation is

$$y(x) = c_1 e^{-3x} \cos 4x + c_2 e^{-3x} \sin 4x = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x)$$

EXAMPLE: Solve the initial-value problem

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \; y'(0) = 4$$

Solution: The characteristic polynomial is

$$P(r) = r^2 + 4r + 4 = r^2 + 2r \cdot 2 + 2^2 = (r+2)^2$$

Thus, $r = -2$ is a repeated root of the characteristic equation, and therefore two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{-2x}, \qquad y_2(x) = xe^{-2x}$$

Consequently, the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} = e^{-2x}(c_1 + c_2 x)$$

The initial condition $y(0) = 1$ implies that

$$y(0) = e^{-2\cdot 0}(c_1 + c_2 \cdot 0) \quad \Longrightarrow \quad 1 = 1 \cdot (c_1 + 0) \quad \Longrightarrow \quad c_1 = 1$$

Thus,

$$y(x) = e^{-2x}(1 + c_2 x)$$

Differentiating this expression yields

$$\begin{aligned} y'(x) &= \Big(e^{-2x}(1 + c_2 x)\Big)' \\ &= (e^{-2x})'(1 + c_2 x) + e^{-2x}(1 + c_2 x)' \\ &= e^{-2x}(-2x)'(1 + c_2 x) + e^{-2x}(1' + (c_2 x)') \\ &= e^{-2x}(-2)(1 + c_2 x) + e^{-2x}(0 + c_2) \\ &= -2e^{-2x}(1 + c_2 x) + c_2 e^{-2x} \end{aligned}$$

therefore the second initial condition implies

$$\begin{aligned} y'(0) &= -2e^{-2\cdot 0}(1 + c_2 \cdot 0) + c_2 e^{-2\cdot 0} \\ 4 &= -2(1 + 0) + c_2 \cdot 1 \\ 4 &= -2 + c_2 \end{aligned}$$

so $c_2 = 6$. Hence, the unique solution to the given initial-value problem is

$$y(x) = e^{-2x}(1 + 6x)$$