# Algebraic Properties of Solutions

We consider first the second-order linear homogeneous equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \tag{1}$$

THEOREM 1 (Existence-Uniqueness Theorem): Let the functions $p(t)$ and $q(t)$ be continuous in the open interval $\alpha < t < \beta$. Then, there exists one, and only one function $y(t)$ satisfying the differential equation (1) on the entire interval $\alpha < t < \beta$, and the prescribed initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

In particular, any solution $y = y(t)$ of (1) which satisfies $y(t_0) = 0$ and $y'(t_0) = 0$ at some time $t = t_0$ must be identically zero.

We begin our analysis of equation (1) with the important observation that the left-hand side

$$y'' + p(t)y' + q(t)y$$

of the differential equation can be viewed as defining a "function of a function": with each function $y$ having two derivatives, we associate another function, which we'll call $L[y]$, by the relation

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

In mathematical terminology, $L$ is an operator which operates on functions; that is, there is a prescribed recipe for associating with each function $y$ a new function $L[y]$.

EXAMPLE: Let $p(t) = 0$ and $q(t) = t$. Then,

$$L[y](t) = y''(t) + ty(t)$$

If $y(t) = \cos t$, then

$$\begin{aligned} L[y](t) &= (\cos t)'' + t\cos t \\ &= (-\sin t)' + t\cos t \\ &= -\cos t + t\cos t \\ &= t\cos t + (-1)\cos t \\ &= (t-1)\cos t \end{aligned}$$

and if $y(t) = t^3$, then

$$\begin{aligned} L[y](t) &= (t^3)'' + t(t^3) \\ &= (3t^2)' + t^4 \\ &= t^4 + 6t \end{aligned}$$

Thus, the operator $L$ assigns the function $(t-1)\cos t$ to the function $\cos t$, and the function $6t + t^4$ to the function $t^3$.

DEFINITION: An operator $L$ which assigns functions to functions and which satisfies the following properties

1. $L[cy] = cL[y]$ for any constant $c$

2. $L[y_1 + y_2] = L[y_1] + L[y_2]$

is called a *linear operator*. All other operators are *nonlinear*.

REMARK: One can show that if $L$ is a linear operator, then

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

An example of a nonlinear operator is $L[y](t) = y(t) + 1$. Indeed,

$$L[y_1 + y_2] = y_1 + y_2 + 1$$

$$L[y_1] + L[y_2] = (y_1 + 1) + (y_2 + 1) = y_1 + y_2 + 2$$

Since

$$L[y_1 + y_2] \neq L[y_1] + L[y_2]$$

it follows that $L[y](t) = y(t) + 1$ is nonlinear.

An example of a linear operator is

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

Indeed,

$$\begin{aligned} L[cy] &= (cy)'' + p(t)(cy)' + q(t)(cy) \\ &= c(y'' + p(t)y' + q(t)y) \\ &= cL[y] \end{aligned}$$

and

$$\begin{aligned} L[y_1 + y_2] &= (y_1 + y_2)'' + p(t)(y_1 + y_2)' + q(t)(y_1 + y_2) \\ &= y_1'' + y_2'' + p(t)y_1' + p(t)y_2' + q(t)y_1 + q(t)y_2 \\ &= y_1'' + p(t)y_1' + q(t)y_1 + y_2'' + p(t)y_2' + q(t)y_2 = L[y_1] + L[y_2] \end{aligned}$$

The usefulness of Properties 1 and 2 lies in the observation that the solutions $y(t)$ of the differential equation (1) are exactly those functions $y$ for which

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

In other words, the solutions $y(t)$ of (1) are exactly those functions $y$ to which the operator $L$ assigns the zero function. Hence, if $y_1(t)$ and $y_2(t)$ are solutions of (1), then

$$c_1 y_1(t) + c_2 y_2(t)$$

is also a solution of (1), since

$$L[c_1 y_1 + c_2 y_2](t) = c_1 L[y_1](t) + c_2 L[y_2](t) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

The preceding argument shows that we can use our knowledge of two solutions $y_1(t)$ and $y_2(t)$ of (1) to generate infinitely many other solutions. Consider, for example, the differential equation

$$\frac{d^2y}{dt^2} + y = 0 \tag{2}$$

Two solutions of (2) are

$$y_1(t) = \cos t \qquad \text{and} \qquad y_2(t) = \sin t$$

Hence,

$$y(t) = c_1 \cos t + c_2 \sin t \tag{3}$$

is also a solution of (2), for every choice of constants $c_1$ and $c_2$. Now, equation (3) contains two arbitrary constants. It is natural to suspect, therefore, that this expression represents the general solution of (2); that is, every solution $y(t)$ of (2) must be of the form (3). This is indeed the case, as we now show. Let $y(t)$ be any solution of (2). By the existence-uniqueness theorem, $y(t)$ exists for all $t$. Let

$$y(0) = y_0, \quad y'(0) = y_0'$$

and consider the function

$$\phi(t) = y_0 \cos t + y_0' \sin t$$

This function is a solution of (2) since it is a linear combination of solutions of (2). Moreover,

$$\phi(0) = y_0 \qquad \text{and} \qquad \phi'(0) = y_0'$$

Thus, $y(t)$ and $\phi(t)$ satisfy the same second-order linear homogeneous equation and the same initial conditions. Therefore, by the uniqueness part of Theorem 1, $y(t)$ must be identically equal to $\phi(t)$, so that

$$y(t) = y_0 \cos t + y_0' \sin t$$

Thus, equation (3) is indeed the general solution of (2).

Let us return now to the general linear equation (1). Suppose, in some manner, that we manage to find two solutions $y_1(t)$ and $y_2(t)$ of (1). Then, every function

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{4}$$

is again a solution of (1). Does the expression (4) represent the general solution of (1)? That is to say, does every solution $y(t)$ of (1) have the form (4)? The following theorem answers this question.

THEOREM 2: Let $y_1(t)$ and $y_2(t)$ be two solutions of (1) on the interval $\alpha < t < \beta$, with

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

unequal to zero in this interval. Then,

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is the general solution of (1).

EXAMPLE: The function $y(t) = c_1 \cos t + c_2 \sin t$ is the general solution of (2), since

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ (\cos t)' & (\sin t)' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = (\cos t)(\cos t) - (\sin t)(-\sin t) = \cos^2 t + \sin^2 t = 1$$

REMARK: In this case, we say that $y_1(t)$ and $y_2(t)$ are a *fundamental set* of solutions of (1).

DEFINITION: The quantity

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is called the *Wronskian* of $y_1$ and $y_2$ and is denoted by $W(t) = W[y_1, y_2](t)$.

DEFINITION: The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly dependent* on an interval $I$ if one of these functions is a constant multiple of the other on $I$. The functions $y_1(t)$ and $y_2(t)$ are said to be linearly independent on an interval $I$ if they are not linearly dependent on $I$.

THEOREM 3: Two solutions $y_1(t)$ and $y_2(t)$ of (1) are linearly independent on the interval $\alpha < t < \beta$ if, and only if, their Wronskian is unequal to zero on this interval.

COROLLARY: Two solutions $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (1) on the interval $\alpha < t < \beta$ if, and only if, they are linearly independent on this interval.

EXAMPLE: The function $y(t) = c_1 \cos t + c_2 \sin t$ is the general solution of (2), since $\cos t$ and $\sin t$ are not constant multiples of each other.