Exact Equations

For the next technique it is best to consider first-order differential equations written in differential form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

where M and N are given functions, assumed to be sufficiently smooth. The method that we will consider is based on the idea of a differential. Recall from a previous calculus course that if $\phi = \phi(x,y)$ is a function of two variables, x and y, then the differential of ϕ, denoted $d\phi$, is defined by

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy \tag{2}$$

EXAMPLE: Solve

$$2x \sin y dx + x^2 \cos y dy = 0 \tag{3}$$

Solution: This equation is separable, but we will use a different technique to solve it. By inspection (see the Appendix, part 1), we notice that

$$2x \sin y dx + x^2 \cos y dy = d(x^2 \sin y)$$

Consequently, equation (3) can be written as

$$d(x^2 \sin y) = 0$$

which implies that $x^2 \sin y$ is constant, hence the general solution to equation (3) is

$$x^2 \sin y = c$$

where c is an arbitrary constant.

In the foregoing example we were able to write the given differential equation in the form $d\phi(x,y) = 0$, and hence obtain its solution. However, we cannot always do this. Indeed we see by comparing equation (1) with (2) that the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

can be written as $d\phi = 0$ if and only if

$$M = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad N = \frac{\partial \phi}{\partial y}$$

for some function ϕ. This motivates the following definition:

DEFINITION: The differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be **exact** in a region R of the xy-plane if there exists a function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = M, \qquad \frac{\partial \phi}{\partial y} = N \tag{4}$$

for all (x,y) in R. Any function ϕ satisfying (4) is called a **potential function** for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

We emphasize that if such a function exists, then the preceding differential equation can be written as

$$d\phi = 0$$

This is why such a differential equation is called an exact differential equation. From the previous example, a potential function for the differential equation

$$2x \sin y dx + x^2 \cos y dy = 0$$

is

$$\phi(x, y) = x^2 \sin y$$

We now show that if a differential equation is exact and we can find a potential function ϕ, its solution can be written down immediately.

THEOREM: The general solution to an exact equation

$$M(x, y)dx + N(x, y)dy = 0$$

is defined implicitly by

$$\phi(x, y) = c$$

where ϕ satisfies (4) and c is an arbitrary constant.

THEOREM (**Test for Exactness**): Let M, N, and their first partial derivatives M_y and N_x, be continuous in a (simply connected) region R of the xy-plane. Then the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact for all x, y in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{5}$$

EXAMPLE: Determine whether the given differential equation is exact.

1. $[1 + \ln(xy)]dx + (x/y)dy = 0$

2. $x^2 y dx - (xy^2 + y^3)dy = 0$

Solution:

1. In this case,

$$M = 1 + \ln(xy), \qquad N = x/y$$

so that (see the Appendix, part 2)

$$M_y = 1/y = N_x$$

It follows from the previous theorem that the differential equation is exact.

2. In this case, we have

$$M = x^2 y, \qquad N = -(xy^2 + y^3)$$

so that (see the Appendix, part 3)

$$M_y = x^2, \quad \text{whereas} \quad N_x = -y^2$$

Since

$$M_y \neq N_x$$

the differential equation is not exact.

EXAMPLE: Find the general solution to

$$2xe^y dx + (x^2e^y + \cos y)dy = 0$$

Solution: We have

$$M(x,y) = 2xe^y, \qquad N(x,y) = x^2e^y + \cos y$$

so that (see the Appendix, part 4)

$$M_y = 2xe^y = N_x$$

Hence the given differential equation is exact, and so there exists a potential function ϕ such that

$$\frac{\partial \phi}{\partial x} = 2xe^y \tag{6}$$

$$\frac{\partial \phi}{\partial y} = x^2e^y + \cos y \tag{7}$$

Integrating equation (6) with respect to x, holding y fixed, yields (see the Appendix, part 5)

$$\phi(x,y) = x^2e^y + h(y) \tag{8}$$

where h is an arbitrary function of y. We now determine $h(y)$ such that (8) also satisfies (7). Taking the derivative of (8) with respect to y yields

$$\frac{\partial \phi}{\partial y} = x^2e^y + \frac{dh}{dy} \tag{9}$$

Equations (7) and (9) give two expressions for $\partial\phi/\partial y$. This allows us to determine h. Subtracting equation (7) from equation (9) gives the consistency requirement

$$\frac{dh}{dy} = \cos y$$

which implies, upon integration, that

$$h(y) = \sin y$$

where we have set the integration constant equal to zero without loss of generality, since we require only one potential function. Substitution into (8) yields the potential function

$$\phi(x,y) = x^2e^y + \sin y$$

Consequently, the given differential equation can be written as

$$d(x^2e^y + \sin y) = 0$$

and so the general solution is

$$x^2e^y + \sin y = c$$

EXAMPLE: Find the general solution to

$$\Big[\sin(xy) + xy\cos(xy) + 2x\Big]dx + \Big[x^2\cos(xy) + 2y\Big]dy = 0$$

EXAMPLE: Find the general solution to

$$\Big[\sin(xy) + xy\cos(xy) + 2x\Big]dx + \Big[x^2\cos(xy) + 2y\Big]dy = 0$$

Solution: We have

$$M(x,y) = \sin(xy) + xy\cos(xy) + 2x \qquad \text{and} \qquad N(x,y) = x^2\cos(xy) + 2y$$

One can check (see the Appendix, part 6) that

$$M_y = 2x\cos(xy) - x^2y\sin(xy) = N_x$$

and so the differential equation is exact. Hence there exists a potential function $\phi(x,y)$ such that

$$\frac{\partial\phi}{\partial x} = \sin(xy) + xy\cos(xy) + 2x \tag{10}$$

$$\frac{\partial\phi}{\partial y} = x^2\cos(xy) + 2y \tag{11}$$

In this case, equation (11) is the simpler equation, and so we integrate it with respect to y, holding x fixed (see the Appendix, part 7), to obtain

$$\phi(x,y) = x\sin(xy) + y^2 + g(x) \tag{12}$$

where $g(x)$ is an arbitrary function of x. We now determine $g(x)$, and hence ϕ, from (10) and (12). Differentiating (12) partially with respect to x (see the Appendix, part 8) yields

$$\frac{\partial\phi}{\partial x} = \sin(xy) + xy\cos(xy) + \frac{dg}{dx} \tag{13}$$

Equations (10) and (13) are consistent if and only if

$$\frac{dg}{dx} = 2x$$

Hence, upon integrating,

$$g(x) = x^2$$

where we have once more set the integration constant to zero without loss of generality, since we require only one potential function. Substituting into (12) gives the potential function

$$\phi(x,y) = x\sin xy + x^2 + y^2$$

The original differential equation can therefore be written as

$$d(x\sin xy + x^2 + y^2) = 0$$

and hence the general solution is

$$x\sin xy + x^2 + y^2 = c$$

Integrating Factors

Usually a given differential equation will not be exact. However, sometimes it is possible to multiply the differential equation by a nonzero function to obtain an exact equation that can then be solved using the technique we have described in this section. Notice that the solution to the resulting exact equation will be the same as that of the original equation, since we multiply by a nonzero function.

DEFINITION: A nonzero function $I(x, y)$ is called an **integrating factor** for the differential equation $M(x, y)dx + N(x, y)dy = 0$ if the differential equation

$$I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0$$

is exact.

EXAMPLE: Show that $I = x^2y$ is an integrating factor for the differential equation

$$(3y^2 + 5x^2y)dx + (3xy + 2x^3)dy = 0 \tag{14}$$

Solution: Multiplying the given differential equation (which is not exact (see the Appendix, part 9)) by x^2y yields

$$(3x^2y^3 + 5x^4y^2)dx + (3x^3y^2 + 2x^5y)dy = 0 \tag{15}$$

One can check (see the Appendix, part 10) that

$$M_y = 9x^2y^2 + 10x^4y = N_x$$

so that the differential equation (15) is exact, and hence $I = x^2y$ is an integrating factor for equation (14). Therefore there exists a potential function $\phi(x, y)$ such that

$$\frac{\partial\phi}{\partial x} = 3x^2y^3 + 5x^4y^2 \tag{16}$$

$$\frac{\partial\phi}{\partial y} = 3x^3y^2 + 2x^5y \tag{17}$$

We integrate (16) it with respect to x, holding y fixed (see the Appendix, part 11), to obtain

$$\phi(x, y) = x^3y^3 + x^5y^2 + h(y) \tag{18}$$

where $h(y)$ is an arbitrary function of y. We now determine $h(y)$, and hence ϕ, from (17) and (18). Differentiating (18) partially with respect to y (see the Appendix, part 12) yields

$$\frac{\partial\phi}{\partial y} = 3x^3y^2 + 2x^5y + \frac{dh}{dy} \tag{19}$$

Equations (17) and (19) are consistent if and only if

$$\frac{dh}{dy} = 0$$

Hence, upon integrating,

$$h(y) = 0$$

where we have once more set the integration constant to zero without loss of generality, since we require only one potential function. Substituting into (18) gives the potential function

$$\phi(x, y) = x^3y^3 + x^5y^2$$

The original differential equation can therefore be written as

$$d(x^3y^3 + x^5y^2) = 0$$

so that the general solution to equation (15) (and hence the general solution to equation (14)) is defined implicitly by

$$x^3y^3 + x^5y^2 = c$$

That is,

$$x^3y^2(y + x^2) = c$$

THEOREM: The function $I(x, y)$ is an integrating factor for

$$M(x, y)dx + N(x, y)dy = 0 \tag{20}$$

if and only if it is a solution to the partial differential equation

$$N\frac{\partial I}{\partial x} - M\frac{\partial I}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) I \tag{21}$$

THEOREM: Consider the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

1. There exists an integrating factor that is dependent only on x if and only if

$$\frac{M_y - N_x}{N} = f(x)$$

a function of x only. In such a case, an integrating factor is

$$I(x) = e^{\int f(x)dx}$$

2. There exists an integrating factor that is dependent only on y if and only if

$$\frac{M_y - N_x}{M} = g(y)$$

a function of y only. In such a case, an integrating factor is

$$I(y) = e^{-\int g(y)dy}$$

EXAMPLE: Solve

$$(2x - y^2)dx + xydy = 0, \quad x > 0 \tag{22}$$

EXAMPLE: Solve

$$(2x - y^2)dx + xydy = 0, \quad x > 0 \tag{22}$$

Solution: The equation is not exact (see the Appendix, part 13). However,

$$\frac{M_y - N_x}{N} = \frac{-2y - y}{xy} = -\frac{3}{x}$$

which is a function of x only. It follows from the preceding theorem that an integrating factor for equation (22) is

$$I(x) = e^{\int f(x)dx} = e^{-\int (3/x)dx} = e^{-3\ln x} = e^{(\ln x)(-3)} = \left(e^{\ln x}\right)^{-3} = x^{-3}$$

Multiplying equation (22) by I yields the exact equation

$$(2x^{-2} - x^{-3}y^2)dx + x^{-2}ydy = 0 \tag{23}$$

(One can check that this is exact (see the Appendix, part 14), although it must be, by the previous theorem.) Hence there exists a potential function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = 2x^{-2} - x^{-3}y^2 \tag{24}$$

$$\frac{\partial \phi}{\partial y} = x^{-2}y \tag{25}$$

In this case, equation (25) is the simpler equation, and so we integrate it with respect to y, holding x fixed (see the Appendix, part 15), to obtain

$$\phi(x, y) = \frac{1}{2}x^{-2}y^2 + g(x) \tag{26}$$

where $g(x)$ is an arbitrary function of x. We now determine $g(x)$, and hence ϕ, from (24) and (26). Differentiating (26) partially with respect to x (see the Appendix, part 16) yields

$$\frac{\partial \phi}{\partial x} = -x^{-3}y^2 + \frac{dg}{dx} \tag{27}$$

Equations (24) and (27) are consistent if and only if

$$\frac{dg}{dx} = 2x^{-2}$$

Hence, upon integrating,

$$g(x) = -2x^{-1}$$

where we have once more set the integration constant to zero without loss of generality, since we require only one potential function. Substituting into (26) gives the potential function

$$\phi(x, y) = \frac{1}{2}x^{-2}y^2 - 2x^{-1}$$

and hence the general solution to (22) is given implicitly by

$$\frac{1}{2}x^{-2}y^2 - 2x^{-1} = c$$

or equivalently,

$$y^2 - 4x = c_1x^2$$

where $c_1 = 2c$.

Appendix

1. We show that

$$(x^2 \sin y)_x = 2x \sin y \qquad \text{and} \qquad (x^2 \sin y)_y = x^2 \cos y$$

We have

$$(x^2 \sin y)_x = \sin y \cdot (x^2)_x = \sin y \cdot (2x) = 2x \sin y$$

and

$$(x^2 \sin y)_y = x^2 \cdot (\sin y)_y = x^2 \cos y$$

2. We show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{1}{y}$$

We have

$$\frac{\partial M}{\partial y} = (1 + \ln(xy))_y = (1)_y + (\ln(xy))_y = 0 + (\ln(xy))_y = (\ln(xy))_y$$

But

$$(\ln u)' = \frac{1}{u} \cdot u' \qquad \text{(This formula also works for partial derivatives)}$$

therefore

$$(\ln(xy))_y = \frac{1}{xy} \cdot (xy)_y = \frac{1}{xy} \cdot x \cdot (y)_y = \frac{1}{xy} \cdot x \cdot 1 = \frac{1}{y}$$

We can get the same result in a different way:

$$(\ln(xy))_y = (\ln x + \ln y)_y = (\ln x)_y + (\ln y)_y = 0 + \frac{1}{y} = \frac{1}{y}$$

We now find $\dfrac{\partial N}{\partial x}$. We have

$$\frac{\partial N}{\partial x} = \left(\frac{x}{y}\right)_x = \left(x \cdot \frac{1}{y}\right)_x = \frac{1}{y} \cdot (x)_x = \frac{1}{y} \cdot 1 = \frac{1}{y}$$

3. We show that

$$\frac{\partial M}{\partial y} = x^2 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -y^2$$

We have

$$\frac{\partial M}{\partial y} = (x^2 y)_y = x^2 \cdot (y)_y = x^2 \cdot 1 = x^2$$

and

$$\begin{aligned}
\frac{\partial N}{\partial x} = (-(xy^2 + y^3))_x &= (-xy^2 - y^3)_x \\
&= (-xy^2)_x - (y^3)_x \\
&= (-xy^2)_x - 0 \\
&= (-xy^2)_x \\
&= y^2 \cdot (-x)_x \\
&= y^2 \cdot (-1) \\
&= -y^2
\end{aligned}$$

4. We show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2xe^y$$

We have

$$\frac{\partial M}{\partial y} = (2xe^y)_y = 2x \cdot (e^y)_y = 2xe^y$$

and

$$\begin{aligned}\frac{\partial N}{\partial x} &= (x^2e^y + \cos y)_x = (x^2e^y)_x + (\cos y)_x \\ &= (x^2e^y)_x + 0 = (x^2e^y)_x = e^y \cdot (x^2)_x = e^y \cdot 2x = 2xe^y\end{aligned}$$

5. We have

$$\int 2xe^y dx = e^y \int 2x dx = x^2e^y + h(y)$$

6. We show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x\cos(xy) - x^2y\sin(xy)$$

We have

$$\begin{aligned}\frac{\partial M}{\partial y} &= (\sin(xy) + xy\cos(xy) + 2x)_y \\ &= (\sin(xy))_y + (xy\cos(xy))_y + (2x)_y \\ &= (\sin(xy))_y + (xy\cos(xy))_y + 0 \\ &= (\sin(xy))_y + (xy\cos(xy))_y\end{aligned} \tag{1}$$

To find $(\sin(xy))_y$, we note that

$$(\sin u)' = \cos u \cdot u' \qquad \text{(This formula also works for partial derivatives)}$$

therefore

$$\begin{aligned}(\sin(xy))_y = \cos(xy) \cdot (xy)_y &= \cos(xy) \cdot x \cdot (y)_y \\ &= \cos(xy) \cdot x \cdot 1 \\ &= x\cos(xy)\end{aligned} \tag{2}$$

To find $(xy\cos(xy))_y$, we first use the Product rule

$$(uv)' = u'v + uv' \qquad \text{(This formula also works for partial derivatives)}$$

and then the Chain rule:

$$\begin{aligned}(xy\cos(xy))_y &= (xy)_y\cos(xy) + xy(\cos(xy))_y \\ &= x \cdot (y)_y \cdot \cos(xy) + xy(-\sin(xy)) \cdot (xy)_y \\ &= x \cdot 1 \cdot \cos(xy) + xy(-\sin(xy)) \cdot x \cdot (y)_y \\ &= x\cos(xy) + xy(-\sin(xy)) \cdot x \cdot 1 \\ &= x\cos(xy) - x^2y\sin(xy)\end{aligned} \tag{3}$$

Using (2) and (3) in (1), we get

$$\frac{\partial M}{\partial y} = (\sin(xy))_y + (xy\cos(xy))_y = x\cos(xy) + x\cos(xy) - x^2y\sin(xy) = 2x\cos(xy) - x^2y\sin(xy)$$

We now find $\dfrac{\partial N}{\partial x}$. We have

$$\begin{aligned}\frac{\partial N}{\partial x} &= (x^2\cos(xy)+2y)_x = (x^2\cos(xy))_x + (2y)_x \\ &= (x^2\cos(xy))_x + 0 = (x^2\cos(xy))_x\end{aligned}$$

To find $(x^2\cos(xy))_x$, we first use the Product rule and then the Chain rule:

$$\begin{aligned}(x^2\cos(xy))_x &= (x^2)_x\cos(xy) + x^2(\cos(xy))_x \\ &= 2x\cos(xy) + x^2(-\sin(xy))_x \cdot (xy)_x \\ &= 2x\cos(xy) + x^2(-\sin(xy))_x \cdot y \cdot (x)_x \\ &= 2x\cos(xy) + x^2(-\sin(xy))_x \cdot y \cdot 1 \\ &= 2x\cos(xy) - x^2y\sin(xy)\end{aligned}$$

7. We have

$$\begin{aligned}\int (x^2\cos(xy)+2y)dy &= \int x^2\cos(xy)dy + \int 2ydy \\ &= x^2\int \cos(xy)dy + y^2 \\ &= x^2 \cdot \frac{1}{x}\sin(xy) + y^2 + g(x) \\ &= x\sin(xy) + y^2 + g(x)\end{aligned}$$

8. We have

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= (x\sin(xy)+y^2+g(x))_x = (x\sin(xy))_x + (y^2)_x + \frac{dg}{dx} \\ &= (x\sin(xy))_x + 0 + \frac{dg}{dx} = (x\sin(xy))_x + \frac{dg}{dx}\end{aligned} \tag{4}$$

To find $(x\sin(xy))_x$, we first use the Product rule and then the Chain rule:

$$\begin{aligned}(x\sin(xy))_x &= (x)_x\sin(xy) + x(\sin(xy))_x \\ &= 1\cdot\sin(xy) + x\cos(xy)\cdot(xy)_x \\ &= 1\cdot\sin(xy) + x\cos(xy)\cdot y\cdot(x)_x \\ &= 1\cdot\sin(xy) + x\cos(xy)\cdot y\cdot 1 \\ &= \sin(xy) + xy\cos(xy)\end{aligned} \tag{5}$$

Using (5) in (4), we get

$$\frac{\partial \phi}{\partial x} = \sin(xy) + xy\cos(xy) + \frac{dg}{dx}$$

9. We have

$$\begin{aligned}(3y^2+5x^2y)_y &= (3y^2)_y+(5x^2y)_y\\ &= 6y+5x^2\cdot(y)_y\\ &= 6y+5x^2\cdot 1\\ &= 6y+5x^2\end{aligned} \qquad \text{and} \qquad \begin{aligned}(3xy+2x^3)_x &= (3xy)_x+(2x^3)_x\\ &= 3y\cdot(x)_x+6x^2\\ &= 3y\cdot 1+6x^2\\ &= 3y+6x^2\end{aligned}$$

so

$$(3y^2+5x^2y)_y \neq (3xy+2x^3)_x$$

10. We show that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=9x^2y^2+10x^4y$$

We have

$$\begin{aligned}\frac{\partial M}{\partial y} &= (3x^2y^3+5x^4y^2)_y\\ &= (3x^2y^3)_y+(5x^4y^2)_y\\ &= 3x^2\cdot(y^3)_y+5x^4\cdot(y^2)_y\\ &= 3x^2\cdot 3y^2+5x^4\cdot 2y\\ &= 9x^2y^2+10x^4y\end{aligned} \qquad \text{and} \qquad \begin{aligned}\frac{\partial N}{\partial x} &= (3x^3y^2+2x^5y)_x\\ &= (3x^3y^2)_x+(2x^5y)_x\\ &= 3y^2\cdot(x^3)_x+2y\cdot(x^5)_x\\ &= 3y^2\cdot 3x^2+2y\cdot 5x^4\\ &= 9x^2y^2+10x^4y\end{aligned}$$

11. We have

$$\begin{aligned}\int(3x^2y^3+5x^4y^2)dx &= \int 3x^2y^3dx+\int 5x^4y^2dx\\ &= 3y^3\int x^2dx+5y^2\int x^4dx\\ &= 3y^3\frac{x^3}{3}+5y^2\frac{x^5}{5}+h(y)\\ &= x^3y^3+x^5y^2+h(y)\end{aligned}$$

12. We have

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= (x^3y^3+x^5y^2+h(y))_y\\ &= (x^3y^3)_y+(x^5y^2)_y+\frac{dh}{dy}\\ &= x^3\cdot(y^3)_y+x^5\cdot(y^2)_y+\frac{dh}{dy}\\ &= 3x^3y^2+2x^5y+\frac{dh}{dy}\end{aligned}$$

13. We have

$$
\begin{aligned}
(2x - y^2)_y &= (2x)_y - (y^2)_y \\
&= 0 - 2y \\
&= -2y
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
(xy)_x &= y \cdot (x)_x \\
&= y \cdot 1 \\
&= y
\end{aligned}
$$

so

$$(2x - y^2)_y \neq (xy)_x$$

14. We show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -2x^{-3}y$$

We have

$$
\begin{aligned}
(2x^{-2} - x^{-3}y^2)_y &= (2x^{-2})_y - (x^{-3}y^2)_y \\
&= 0 - x^{-3} \cdot (y^2)_y \\
&= -2x^{-3}y
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
(x^{-2}y)_x &= y \cdot (x^{-2})_x \\
&= y \cdot (-2)x^{-3} \\
&= -2x^{-3}y
\end{aligned}
$$

15. We have

$$
\begin{aligned}
\int (x^{-2}y)dy &= x^{-2} \int ydy \\
&= \frac{1}{2}x^{-2}y^2 + g(x)
\end{aligned}
$$

16. We have

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \left(\frac{1}{2}x^{-2}y^2 + g(x)\right)_x \\
&= \left(\frac{1}{2}x^{-2}y^2\right)_x + \frac{dg}{dx} \\
&= \frac{1}{2}y^2 \cdot (x^{-2})_x + \frac{dg}{dx} \\
&= \frac{1}{2}y^2 \cdot (-2)x^{-3} + \frac{dg}{dx} \\
&= -x^{-3}y^2 + \frac{dg}{dx}
\end{aligned}
$$