Separable Equations

A **separable equation** is a first-order differential equation that can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. Equivalently, if $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \tag{1}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)dy = g(x)dx$$

so that all y's are on one side of the equation and all x's are on the other side. Then we integrate both sides of the equation:

$$\int h(y)dy = \int g(x)dx \tag{2}$$

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

We use the Chain Rule to justify this procedure: If h and g satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)dy\right) = \frac{d}{dx}\left(\int g(x)dx\right)$$

so

$$\frac{d}{dy}\left(\int h(y)dy\right)\frac{dy}{dx} = g(x)$$

and

$$h(y)\frac{dy}{dx} = g(x)$$

EXAMPLE:

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

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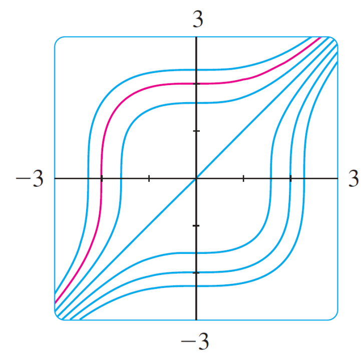
Solution:

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$



hence

$$y = \sqrt[3]{x^3 + 3C}$$

which can be rewritten as

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$.

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$. Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

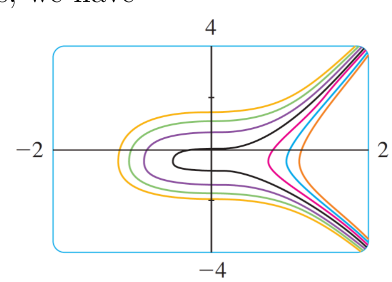
EXAMPLE: Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Solution: Writing the equation in differential form and integrating both sides, we have

$$(2y + \cos y)dy = 6x^2 dx$$

$$\int (2y + \cos y)dy = \int 6x^2 dx$$

$$y^2 + \sin y = 2x^3 + C$$



where C is a constant. The last equation gives the general solution implicitly. In this case it's impossible to solve the equation to express y explicitly as a function of x.

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Solution: First we rewrite the equation using Leibniz notation:

$$\frac{dy}{dx} = x^2 y$$

If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = x^2 dx \qquad y \neq 0$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln |y| = \frac{x^3}{3} + C$$

This equation defines y implicitly as a function of x. But in this case we can solve explicitly for y as follows:

$$|y| = e^{\ln |y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$

so

$$y = \pm e^C e^{x^3/3}$$

We can easily verify that the function $y = 0$ is also a solution of the given differential equation. So we can write the general solution in the form

$$y = A e^{x^3/3}$$

where A is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$).