First-Order Linear Differential Equations

DEFINITION. The general first-order linear differential equation is

$$\frac{dy}{dt} + a(t)y = b(t) \tag{1}$$

Unless otherwise stated, the functions $a(t)$ and $b(t)$ are assumed to be continuous functions of time. The equation

$$\frac{dy}{dt} + a(t)y = 0 \tag{2}$$

is called the *homogeneous* first-order linear differential equation, and Equation (1) is called the *nonhomogeneous* first-order linear differential equation for $b(t)$ not identically zero.

Fortunately, the homogeneous equation (2) can be solved quite easily. First, divide both sides of the equation by y and rewrite it in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t)$$

Second, observe that

$$\frac{\frac{dy}{dt}}{y} \equiv \frac{d}{dt} \ln |y(t)|$$

where by $\ln |y(t)|$ we mean the natural logarithm of $|y(t)|$. Hence Equation (2) can be written in the form

$$\frac{d}{dt} \ln |y(t)| = -a(t)$$

Integrating both sides yields

$$\ln |y(t)| = -\int a(t)dt + c_1$$

where c_1 is an arbitrary constant of integration. Taking exponentials of both sides yields

$$\begin{aligned} |y(t)| &= \exp\left(-\int a(t)dt + c_1\right) \\ &= \exp\left(-\int a(t)dt\right) \cdot \exp(c_1) \\ &= c_2 \exp\left(-\int a(t)dt\right) \end{aligned}$$

where

$$c_2 = \exp(c_1)$$

is a positive constant. From this it follows that

$$y(t) = \pm c_2 \exp\left(-\int a(t)dt\right)$$

Note that $y(t)=0$ is a solution of (2), therefore the *general* solution of (2) is

$$y(t) = c\exp\left(-\int a(t)dt\right) \tag{3}$$

where c is *any* real number.

EXAMPLE: Find the general solution of the equation

$$\frac{dy}{dt} + 2ty = 0 \tag{4}$$

Solution 1: By (3) with $a(t)=2t$ we have

$$y(t) = c\exp\left(-\int 2tdt\right) = ce^{-t^2}$$

where c is any real number.

Solution 2: If $y(t) \neq 0$, then

$$\begin{aligned}
\frac{dy}{dt} + 2ty &= 0 \\
\frac{dy}{dt} &= -2ty \\
dy &= -2tydt \\
\frac{1}{y}dy &= -2tdt \\
\int \frac{1}{y}dy &= -\int 2tdt \\
\ln|y| &= -t^2 + c_1 \\
e^{\ln|y|} &= e^{-t^2+c_1} \\
|y| &= e^{-t^2+c_1} \\
&= e^{-t^2}e^{c_1} \\
&= c_2 e^{-t^2}
\end{aligned}$$

or, in short,

$$\begin{aligned}
\frac{dy}{dt} + 2ty &= 0 \\
\frac{1}{y}dy &= -2tdt \\
\int \frac{1}{y}dy &= -\int 2tdt \\
\ln|y| &= -t^2 + c_1 \\
|y| &= c_2 e^{-t^2}
\end{aligned}$$

where

$$c_2 = e^{c_1}$$

is a positive constant. From this it follows that

$$y = \pm c_2 e^{-t^2}$$

Note that $y(t)=0$ is a solution of (4), therefore the general solution of (4) is

$$y(t) = ce^{-t^2}$$

where c is any real number.

In applications, we are usually not interested in all solutions of (2). Rather, we are looking for the *specific* solution $y(t)$ which at some initial time t_0 has the value y_0. Thus, we want to determine a function $y(t)$ such that

$$\frac{dy}{dt} + a(t)y = 0, \qquad y(t_0) = y_0$$

One can show that

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} a(s)ds \right) \tag{5}$$

EXAMPLE: Find the solution of the initial-value problem

$$\frac{dy}{dt} + (\sin t)y = 0, \qquad y(0) = \frac{3}{2}$$

Solution 1: By (5) with $a(t) = \sin t,\ t_0 = 0$ and $y_0 = \dfrac{3}{2}$ we have

$$y(t) = \frac{3}{2}\exp\left(-\int_0^t \sin s ds \right) = \frac{3}{2}\exp\left(-(-\cos s)\Big|_0^t \right) = \frac{3}{2}\exp\left(\cos s\Big|_0^t \right) = \frac{3}{2}e^{\cos t - \cos 0} = \frac{3}{2}e^{\cos t - 1}$$

Solution 2: We have

$$\frac{dy}{dt} + (\sin t)y = 0$$

$$\begin{aligned}
\frac{dy}{dt} &= -(\sin t)y \\
dy &= -(\sin t)y dt \\
\frac{1}{y}dy &= -\sin t dt \\
\int \frac{1}{y}dy &= -\int \sin t dt \\
\ln|y| &= \cos t + c_1 \\
e^{\ln|y|} &= e^{\cos t + c_1} \\
|y| &= e^{\cos t + c_1} = e^{\cos t}e^{c_1} = c_2 e^{\cos t}
\end{aligned}$$

or, in short,

$$\begin{aligned}
\frac{dy}{dt} + (\sin t)y &= 0 \\
\frac{1}{y}dy &= -\sin t dt \\
\int \frac{1}{y}dy &= -\int \sin t dt \\
\ln|y| &= \cos t + c_1 \\
|y| &= c_2 e^{\cos t}
\end{aligned}$$

where $c_2 = e^{c_1}$ is a positive constant. Since $y(0) = \dfrac{3}{2}$, we get

$$\frac{3}{2} = c_2 e^{\cos 0} = c_2 e^1 = c_2 e \qquad \Longrightarrow \qquad c_2 = \frac{3}{2}e^{-1}$$

so

$$|y| = c_2 e^{\cos t} = \frac{3}{2}e^{-1}e^{\cos t} = \frac{3}{2}e^{\cos t - 1}$$

Note that $y \neq -\dfrac{3}{2}e^{\cos t - 1}$, since $y(0) = \dfrac{3}{2}$. Therefore $y = \dfrac{3}{2}e^{\cos t - 1}$.

EXAMPLE: Find the solution of the initial-value problem

$$\frac{dy}{dt} + e^{t^2} y = 0, \qquad y(1) = 2$$

Solution: By (5) with $a(t) = e^{t^2}, t_0 = 1$ and $y_0 = 2$ we have

$$y(t) = 2 \exp \left(- \int_1^t e^{s^2} ds \right)$$

We return now to the nonhomogeneous equation

$$\frac{dy}{dt} + a(t)y = b(t)$$

It should be clear from our analysis of the homogeneous equation that the way to solve the nonhomogeneous equation is to express it in the form

$$\frac{d}{dt}(\text{“something”}) = b(t)$$

and then to integrate both sides to solve for “something”. However, the expression

$$\frac{dy}{dt} + a(t)y$$

does not appear to be the derivative of some simple expression. The next logical step in our analysis therefore should be the following: Can we make the left hand side of the equation to be d/dt of “something”? More precisely, we can multiply both sides of (1) by any continuous function $\mu(t)$ to obtain the equivalent equation

$$\mu(t)\frac{dy}{dt} + a(t)\mu(t)y = \mu(t)b(t) \tag{6}$$

(By equivalent equations we mean that every solution of (6) is a solution of (1) and vice-versa.) Thus, can we *choose* $\mu(t)$ so that

$$\mu(t)\frac{dy}{dt} + a(t)\mu(t)y$$

is the derivative of some simple expression? The answer to this question is yes, and is obtained by observing that

$$\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

Hence, the left-hand side of (6) will be equal to the derivative of $\mu(t)y$ if and only if

$$\mu(t)\frac{dy}{dt} + a(t)\mu(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

$$a(t)\mu(t)y = \frac{d\mu}{dt}y$$

$$\frac{d\mu}{dt} = a(t)\mu(t)$$

But this is a first-order linear homogeneous equation for $\mu(t)$, i.e.

$$\frac{d\mu}{dt} - a(t)\mu(t) = 0$$

which we already know how to solve, and since we only need one such function $\mu(t)$ we set the constant c in (3) equal to one and take

$$\mu(t) = \exp\left(-\int(-a(t))dt\right) = \exp\left(\int a(t)dt\right)$$

The function $\mu(t)$ is called an *integrating factor* for the nonhomogeneous equation. For this $\mu(t)$, Equation (6) can be written as

$$\frac{d}{dt}(\mu(t)y) = \mu(t)b(t) \tag{7}$$

To obtain the general solution of the nonhomogeneous equation (1), that is, to find all solutions of the nonhomogeneous equation, we take the indefinite integral (anti-derivative) of both sides of (7) which yields

$$\mu(t)y = \int \mu(t)b(t)dt + c$$

or

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)dt + c\right) = \exp\left(-\int a(t)dt\right)\left(\int \mu(t)b(t)dt + c\right)$$

Alternately, if we are interested in the specific solution of (1) satisfying the initial condition $y(t_0) = y_0$, that is, if we want to solve the initial-value problem

$$\frac{dy}{dt} + a(t)y = b(t), \qquad y(t_0) = y_0$$

then we can take the definite integral of both sides of (7) between t_0 and t to obtain that

$$\mu(t)y - \mu(t_0)y_0 = \int_{t_0}^{t} \mu(s)b(s)ds$$

or

$$y = \frac{1}{\mu(t)}\left(\mu(t_0)y_0 + \int_{t_0}^{t} \mu(s)b(s)ds\right)$$

EXAMPLE: Find the general solution of the equation

$$\frac{dy}{dt} - 2ty = t \tag{8}$$

Solution: Here $a(t) = -2t$ so that

$$\mu(t) = \exp\left(\int a(t)dt\right) = \exp\left(-\int 2tdt\right) = e^{-t^2}$$

Multiplying both sides of equation (8) by $\mu(t)$ we obtain the equivalent equation

$$e^{-t^2}\left(\frac{dy}{dt}-2ty\right)=te^{-t^2} \qquad \text{or} \qquad \frac{d}{dt}\left(e^{-t^2}y\right)=te^{-t^2}$$

Hence,

$$e^{-t^2}y=\int te^{-t^2}dt=\begin{bmatrix} -t^2=u \\ d(-t^2)=du \\ -2tdt=du \\ tdt=-\frac{1}{2}du \end{bmatrix}=-\frac{1}{2}\int e^u du=-\frac{1}{2}e^u+c=-\frac{e^{-t^2}}{2}+c$$

and

$$y(t)=e^{t^2}\left(-\frac{e^{-t^2}}{2}+c\right)=-\frac{1}{2}+ce^{t^2}$$

EXAMPLE: Find the solution of the initial-value problem

$$\frac{dy}{dt}+2ty=t, \qquad y(1)=2 \tag{9}$$

Solution: Here $a(t)=2t$ so that

$$\mu(t)=\exp\left(\int a(t)dt\right)=\exp\left(\int 2tdt\right)=e^{t^2}$$

Multiplying both sides of equation (9) by $\mu(t)$ we obtain the equivalent equation

$$e^{t^2}\left(\frac{dy}{dt}+2ty\right)=te^{t^2} \qquad \text{or} \qquad \frac{d}{dt}\left(e^{t^2}y\right)=te^{t^2}$$

Hence,

$$\int_1^t \frac{d}{ds}\left(e^{s^2}y(s)\right)ds=\int_1^t se^{s^2}ds$$

so that

$$e^{s^2}y(s)\Big|_1^t=\frac{e^{s^2}}{2}\Big|_1^t$$

$$e^{t^2}y(t)-e^{1^2}y(1)=\frac{e^{t^2}}{2}-\frac{e^{1^2}}{2}$$

$$e^{t^2}y(t)-2e=\frac{e^{t^2}}{2}-\frac{e}{2}$$

$$e^{t^2}y(t)=\frac{e^{t^2}}{2}+\frac{3e}{2}$$

therefore

$$y(t)=\frac{1}{2}+\frac{3e}{2}e^{-t^2}=\frac{1}{2}+\frac{3}{2}e^{1-t^2}$$

EXAMPLE: Find the solution of the initial-value problem

$$\frac{dy}{dx} + xy = xe^{x^2/2}, \qquad y(0) = 1 \tag{10}$$

Solution: An appropriate integrating factor in this case is

$$\mu(x) = e^{\int x dx} = e^{x^2/2}$$

Multiplying both sides of equation (10) by $\mu(x)$ we obtain the equivalent equation

$$\frac{dy}{dx} + xy = xe^{x^2/2}$$

$$\frac{dy}{dx}e^{x^2/2} + xye^{x^2/2} = xe^{x^2/2}e^{x^2/2}$$

$$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2}$$

Integrating both sides with respect to x, we obtain

$$e^{x^2/2}y = \frac{1}{2}e^{x^2} + c$$

Hence,

$$y(x) = e^{-x^2/2}\left(\frac{1}{2}e^{x^2} + c\right)$$

Imposing the initial condition $y(0) = 1$ yields

$$1 = \frac{1}{2} + c$$

so that $c = \frac{1}{2}$. Thus the required particular solution is

$$y(x) = \frac{1}{2}e^{-x^2/2}(e^{x^2} + 1) = \frac{1}{2}(e^{x^2/2} + e^{-x^2/2})$$

EXAMPLE: Solve

$$x\frac{dy}{dx} + 2y = \cos x, \;\; x > 0$$

EXAMPLE: Solve

$$x\frac{dy}{dx} + 2y = \cos x,\ \ x > 0$$

Solution: We first write the given differential equation in standard form. Dividing by x yields

$$\frac{dy}{dx} + 2x^{-1}y = x^{-1}\cos x \tag{11}$$

An integrating factor is

$$\mu(x) = e^{\int 2x^{-1}dx} = e^{2\ln x} = (e^{\ln x})^2 = x^2$$

Multiplying both sides of equation (11) by $\mu(x)$ we obtain the equivalent equation

$$\frac{d}{dx}(x^2y) = x\cos x$$

Hence,

$$x^2y = \int x\cos x dx$$

By the integration by parts formula

$$\int u dv = uv - \int v du$$

we get

$$\int x\cos x dx = \left[\begin{array}{c|c} x = u & \cos x dx = dv \\ & \\ dx = du & \sin x = v \end{array}\right] = x\sin x - \int \sin x dx = x\sin x - (-\cos x) + c$$

$$= x\sin x + \cos x + c$$

therefore

$$x^2y(x) = x\sin x + \cos x + c$$

so

$$y(x) = x^{-2}(x\sin x + \cos x + c)$$