The Existence-Uniqueness Theorem

Consider the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

where f is a given function of t and y. One can show that $y(t)$ is a solution of (1) if, and only if, it is a continuous solution of

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{2}$$

Consider the following sequence of functions $y_n(t)$:

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds \tag{3}$$

These functions $y_n(t)$ are called successive approximations, or Picard iterates. Remarkably, these Picard iterates always converge, on a suitable interval, to a solution $y(t)$ of (2).

EXAMPLE: Compute the Picard iterates for the initial-value problem

$$y' = y, \quad y(0) = 1$$

and show that they converge to the solution $y(t) = e^t$.

Solution: The integral equation corresponding to this initial-value problem is

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds = 1 + \int_{0}^{t} y(s)ds \quad \Longrightarrow \quad y_{n+1}(t) = 1 + \int_{0}^{t} y_n(s)ds$$

Hence,

$$y_0(t) = y(0) = 1$$

$$y_1(t) = 1 + \int_{0}^{t} y_0(s)ds = 1 + \int_{0}^{t} 1ds = 1 + s\Big|_0^t = 1 + t$$

$$y_2(t) = 1 + \int_{0}^{t} y_1(s)ds = 1 + \int_{0}^{t} (1+s)ds = 1 + \left(s + \frac{s^2}{2}\right)\Bigg|_0^t = 1 + t + \frac{t^2}{2} = 1 + t + \frac{t^2}{2!}$$

$$y_3(t) = 1 + \int_{0}^{t} y_2(s)ds = 1 + \int_{0}^{t} \left(1 + s + \frac{s^2}{2!}\right) ds = 1 + \left(s + \frac{s^2}{2} + \frac{s^3}{2! \cdot 3}\right)\Bigg|_0^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

and, in general,

$$y_n(t) = 1 + \int_{0}^{t} y_{n-1}(s)ds = 1 + \int_{0}^{t} \left(1 + s + \ldots + \frac{s^{n-1}}{(n-1)!}\right) ds$$

$$= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!}$$

Since

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$$

we see that the Picard iterates $y_n(t)$ converge to the solution $y(t)$ of this initial-value problem.

EXAMPLE: Compute the Picard iterates $y_1(t), y_2(t)$ for the initial-value problem

$$y' = 1 + y^3, \quad y(1) = 1$$

Solution: The integral equation corresponding to this initial-value problem is

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds = 1 + \int_{1}^{t} [1 + y^3(s)]ds \quad \Longrightarrow \quad y_{n+1}(t) = 1 + \int_{1}^{t} [1 + y_n^3(s)]ds$$

Hence,

$$y_0(t) = y(1) = 1$$

$$y_1(t) = 1 + \int_{1}^{t} [1 + y_0^3(s)]ds = 1 + \int_{1}^{t} (1+1)ds = 1 + 2s\Big|_1^t = 1 + 2(t-1)$$

$$y_2(t) = 1 + \int_{1}^{t} [1 + y_1^3(s)]ds = 1 + \int_{1}^{t} \left\{1 + [1 + 2(s-1)]^3\right\} ds$$

$$= 1 + 2(t-1) + 3(t-1)^2 + 4(t-1)^3 + 2(t-1)^4 \qquad \text{(see Appendix)}$$

Notice that it is already quite cumbersome to compute $y_3(t)$.

THEOREM: Let f and $\partial f/\partial y$ be continuous in the rectangle R:

$$t_0 \le t \le t_0 + a, \qquad |y - y_0| \le b$$

Compute

$$M = \max_{(t,y)\in R} |f(t, y)|$$

and set

$$\alpha = \min\left(a, \frac{b}{M}\right)$$

Then, the initial-value problem (1)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

has a unique solution $y(t)$ on the interval $t_0 \le t \le t_0 + \alpha$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \quad y(0) = 0$$

exists for $0 \le t \le \frac{1}{2}$, and in this interval, $|y(t)| \le 1$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \qquad y(0) = 0$$

exists for $0 \le t \le \dfrac{1}{2}$, and in this interval, $|y(t)| \le 1$.

Solution: Let R be the rectangle $0 \le t \le \dfrac{1}{2}$, $|y(t)| \le 1$. Computing

$$\begin{aligned} M &= \max_{(t,y)\in R} |f(t,y)| \\ &= \max_{(t,y)\in R} (t^2 + e^{-y^2}) = \left(\frac{1}{2}\right)^2 + e^{-0^2} = \frac{1}{4} + 1 = \frac{5}{4} \end{aligned}$$

we see that $y(t)$ exists for

$$t_0 \le t \le t_0 + \alpha$$

$$t_0 \le t \le t_0 + \min\left(a, \frac{b}{M}\right)$$

$$\left[\begin{array}{lll} \text{WORK :} & & \\ y(t_0) = y_0 & \stackrel{y(0)=0}{\Longrightarrow} & t_0 = 0 \\ t_0 \le t \le t_0 + a & \stackrel{0\le t\le 0+(1/2)}{\Longrightarrow} & a = \dfrac{1}{2} \\ |y - y_0| \le b & \stackrel{|y(t)-0|\le 1}{\Longrightarrow} & b = 1 \end{array}\right]$$

$$0 \le t \le 0 + \min\left(\frac{1}{2}, \frac{1}{5/4}\right) = \frac{1}{2}$$

and in this interval, $|y(t)| \le 1$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \qquad y(0) = 1$$

exists for $0 \le t \le \dfrac{1}{9}$, and in this interval, $0 \le y \le 2$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \qquad y(0) = 1$$

exists for $0 \le t \le \frac{1}{9}$, and in this interval, $0 \le y \le 2$.

Solution: Let R be the rectangle $0 \le t \le \frac{1}{9}$, $0 \le y \le 2$. Computing

$$\begin{aligned} M &= \max_{(t,y)\in R} |f(t,y)| \\ &= \max_{(t,y)\in R} (e^{-t^2} + y^3) = e^{-0^2} + 2^3 = 1 + 8 = 9 \end{aligned}$$

we see that $y(t)$ exists for

$$t_0 \le t \le t_0 + \alpha$$

$$t_0 \le t \le t_0 + \min\left(a, \frac{b}{M}\right)$$

$$\left[\begin{array}{l} \text{WORK :} \\ y(t_0) = y_0 \quad \overset{y(0)=1}{\Longrightarrow} \quad t_0 = 0 \\ \\ t_0 \le t \le t_0 + a \quad \overset{0\le t\le 0+(1/9)}{\Longrightarrow} \quad a = \dfrac{1}{9} \\ \\ |y - y_0| \le b \quad \Longrightarrow \quad \Big[0 \le y \le 2 \ \Rightarrow\ 0 - 1 \le y - 1 \le 2 - 1 \ \Rightarrow\ |y-1| \le 1\Big] \quad \Longrightarrow \quad b = 1 \end{array}\right]$$

$$0 \le t \le 0 + \min\left(\frac{1}{9}, \frac{1}{9}\right) = \frac{1}{9}$$

and in this interval, $0 \le y \le 2$.

Appendix

EXAMPLE: Find $\int\limits_1^t \left\{1+[1+2(s-1)]^3\right\} ds$.

Solution 1: Since $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$, we have

$$\begin{aligned}[1+2(s-1)]^3 &= 1^3 + 3\cdot 1^2 \cdot 2(s-1) + 3\cdot 1\cdot [2(s-1)]^2 + [2(s-1)]^3 \\ &= 1+6(s-1)+12(s-1)^2+8(s-1)^3\end{aligned}$$

therefore

$$\begin{aligned}\int\limits_1^t \left\{1+[1+2(s-1)]^3\right\} ds &= \int\limits_1^t \left\{1+1+6(s-1)+12(s-1)^2+8(s-1)^3\right\} ds \\ &= \int\limits_1^t \left\{2+6(s-1)+12(s-1)^2+8(s-1)^3\right\} ds \\ &= \int\limits_1^t 2ds + \int\limits_1^t 6(s-1)ds + \int\limits_1^t 12(s-1)^2 ds + \int\limits_1^t 8(s-1)^3 ds \\ &= 2s\Big|_1^t + 6\frac{(s-1)^2}{2}\Big|_1^t + 12\frac{(s-1)^3}{3}\Big|_1^t + 8\frac{(s-1)^4}{4}\Big|_1^t \\ &= 2s\Big|_1^t + 3(s-1)^2\Big|_1^t + 4(s-1)^3\Big|_1^t + 2(s-1)^4\Big|_1^t \\ &= 2(t-1)+3(t-1)^2+4(t-1)^3+2(t-1)^4\end{aligned}$$

Solution 2: We have

$$\begin{aligned}\int\limits_1^t \left\{1+[1+2(s-1)]^3\right\} ds = \left[\begin{array}{l} 1+2(s-1)=u \\ d(1+2(s-1))=du \\ 2ds=du \\ ds=\frac{1}{2}du \end{array}\right] &= \frac{1}{2}\int\limits_{1+2(1-1)}^{1+2(t-1)} (1+u^3)du \\ &= \frac{1}{2}\int\limits_1^{2t-1} (1+u^3)du \\ &= \frac{1}{2}\left(u+\frac{u^4}{4}\right)\Big|_1^{2t-1} \\ &= \frac{1}{2}\left[\left(2t-1+\frac{(2t-1)^4}{4}\right)-\left(1+\frac{1^4}{4}\right)\right] \\ &= \frac{1}{2}\left(2t-\frac{9}{4}+\frac{(2t-1)^4}{4}\right)\end{aligned}$$

One can check that $\frac{1}{2}\left(2t-\frac{9}{4}+\frac{(2t-1)^4}{4}\right)$ is identically equal to $2(t-1)+3(t-1)^2+4(t-1)^3+2(t-1)^4$.