

The Existence-Uniqueness Theorem

Consider the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \quad (1)$$

where f is a given function of t and y . One can show that $y(t)$ is a solution of (1) if, and only if, it is a continuous solution of

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds \quad (2)$$

Consider the following sequence of functions $y_n(t)$:

$$y_{n+1}(t) = y_0 + \int_{t_0}^t f(s, y_n(s)) ds \quad (3)$$

These functions $y_n(t)$ are called successive approximations, or Picard iterates. Remarkably, these Picard iterates always converge, on a suitable interval, to a solution $y(t)$ of (2).

EXAMPLE: Compute the Picard iterates for the initial-value problem

$$y' = y, \quad y(0) = 1$$

and show that they converge to the solution $y(t) = e^t$.

Solution: The integral equation corresponding to this initial-value problem is

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds = 1 + \int_0^t y(s) ds \implies y_{n+1}(t) = 1 + \int_0^t y_n(s) ds$$

Hence,

$$y_0(t) = y_0 = 1$$

$$y_1(t) = 1 + \int_0^t y_0(s) ds = 1 + \int_0^t 1 ds = 1 + s \Big|_0^t = 1 + t$$

$$y_2(t) = 1 + \int_0^t y_1(s) ds = 1 + \int_0^t (1 + s) ds = 1 + \left(s + \frac{s^2}{2} \right) \Big|_0^t = 1 + t + \frac{t^2}{2} = 1 + t + \frac{t^2}{2!}$$

$$y_3(t) = 1 + \int_0^t y_2(s) ds = 1 + \int_0^t \left(1 + s + \frac{s^2}{2!} \right) ds = 1 + \left(s + \frac{s^2}{2} + \frac{s^3}{2! \cdot 3} \right) \Big|_0^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

and, in general,

$$\begin{aligned} y_n(t) &= 1 + \int_0^t y_{n-1}(s) ds = 1 + \int_0^t \left(1 + s + \dots + \frac{s^{n-1}}{(n-1)!} \right) ds \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^n}{n!} \end{aligned}$$

Since

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

we see that the Picard iterates $y_n(t)$ converge to the solution $y(t)$ of this initial-value problem.

EXAMPLE: Compute the Picard iterates $y_1(t), y_2(t)$ for the initial-value problem

$$y' = 1 + y^3, \quad y(1) = 1$$

Solution: The integral equation corresponding to this initial-value problem is

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds = 1 + \int_1^t [1 + y^3(s)] ds \implies y_{n+1}(t) = 1 + \int_1^t [1 + y_n^3(s)] ds$$

Hence,

$$y_0(t) = y_0 = 1$$

$$y_1(t) = 1 + \int_1^t [1 + y_0^3(s)] ds = 1 + \int_1^t (1 + 1) ds = 1 + 2s \Big|_1^t = 1 + 2(t - 1)$$

$$\begin{aligned} y_2(t) &= 1 + \int_1^t [1 + y_1^3(s)] ds = 1 + \int_1^t \{1 + [1 + 2(s - 1)]^3\} ds \\ &= 1 + 2(t - 1) + 3(t - 1)^2 + 4(t - 1)^3 + 2(t - 1)^4 \quad (\text{see Appendix}) \end{aligned}$$

Notice that it is already quite cumbersome to compute $y_3(t)$.

THEOREM: Let f and $\partial f/\partial y$ be continuous in the rectangle R :

$$t_0 \leq t \leq t_0 + a, \quad |y - y_0| \leq b$$

Compute

$$M = \max_{(t,y) \in R} |f(t, y)|$$

and set

$$\alpha = \min \left(a, \frac{b}{M} \right)$$

Then, the initial-value problem (1)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

has a unique solution $y(t)$ on the interval $t_0 \leq t \leq t_0 + \alpha$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \quad y(0) = 0$$

exists for $0 \leq t \leq \frac{1}{2}$, and in this interval, $|y(t)| \leq 1$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \quad y(0) = 0$$

exists for $0 \leq t \leq \frac{1}{2}$, and in this interval, $|y(t)| \leq 1$.

Solution: Let R be the rectangle $0 \leq t \leq \frac{1}{2}$, $|y(t)| \leq 1$. Computing

$$\begin{aligned} M &= \max_{(t,y) \in R} |f(t,y)| \\ &= \max_{(t,y) \in R} (t^2 + e^{-y^2}) = \left(\frac{1}{2}\right)^2 + e^{-0^2} = \frac{1}{4} + 1 = \frac{5}{4} \end{aligned}$$

we see that $y(t)$ exists for

$$t_0 \leq t \leq t_0 + \alpha$$

$$t_0 \leq t \leq t_0 + \min\left(a, \frac{b}{M}\right)$$

$$\left[\begin{array}{l} \text{WORK :} \\ y(t_0) = y_0 \quad \xrightarrow{y(0)=0} \quad t_0 = 0 \\ \\ t_0 \leq t \leq t_0 + a \quad \xrightarrow{0 \leq t \leq 0 + (1/2)} \quad a = \frac{1}{2} \\ \\ |y - y_0| \leq b \quad \xrightarrow{|y(t)-0| \leq 1} \quad b = 1 \end{array} \right]$$

$$0 \leq t \leq 0 + \min\left(\frac{1}{2}, \frac{1}{5/4}\right) = \frac{1}{2}$$

and in this interval, $|y(t)| \leq 1$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \quad y(0) = 1$$

exists for $0 \leq t \leq \frac{1}{9}$, and in this interval, $0 \leq y \leq 2$.

EXAMPLE: Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \quad y(0) = 1$$

exists for $0 \leq t \leq \frac{1}{9}$, and in this interval, $0 \leq y \leq 2$.

Solution: Let R be the rectangle $0 \leq t \leq \frac{1}{9}$, $0 \leq y \leq 2$. Computing

$$\begin{aligned} M &= \max_{(t,y) \in R} |f(t,y)| \\ &= \max_{(t,y) \in R} (e^{-t^2} + y^3) = e^{-0^2} + 2^3 = 1 + 8 = 9 \end{aligned}$$

we see that $y(t)$ exists for

$$t_0 \leq t \leq t_0 + \alpha$$

$$t_0 \leq t \leq t_0 + \min\left(a, \frac{b}{M}\right)$$

$$\left[\begin{array}{l} \text{WORK :} \\ y(t_0) = y_0 \quad \xrightarrow{y(0)=1} \quad t_0 = 0 \\ t_0 \leq t \leq t_0 + a \quad \xrightarrow{0 \leq t \leq 0+(1/9)} \quad a = \frac{1}{9} \\ |y - y_0| \leq b \quad \implies \quad [0 \leq y \leq 2 \implies 0 - 1 \leq y - 1 \leq 2 - 1 \implies |y - 1| \leq 1] \quad \implies \quad b = 1 \end{array} \right]$$

$$0 \leq t \leq 0 + \min\left(\frac{1}{9}, \frac{1}{9}\right) = \frac{1}{9}$$

and in this interval, $0 \leq y \leq 2$.

Appendix

EXAMPLE: Find $\int_1^t \{1 + [1 + 2(s - 1)]^3\} ds$.

Solution 1: Since $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we have

$$\begin{aligned} [1 + 2(s - 1)]^3 &= 1^3 + 3 \cdot 1^2 \cdot 2(s - 1) + 3 \cdot 1 \cdot [2(s - 1)]^2 + [2(s - 1)]^3 \\ &= 1 + 6(s - 1) + 12(s - 1)^2 + 8(s - 1)^3 \end{aligned}$$

therefore

$$\begin{aligned} \int_1^t \{1 + [1 + 2(s - 1)]^3\} ds &= \int_1^t \{1 + 1 + 6(s - 1) + 12(s - 1)^2 + 8(s - 1)^3\} ds \\ &= \int_1^t \{2 + 6(s - 1) + 12(s - 1)^2 + 8(s - 1)^3\} ds \\ &= \int_1^t 2ds + \int_1^t 6(s - 1)ds + \int_1^t 12(s - 1)^2ds + \int_1^t 8(s - 1)^3ds \\ &= 2s \Big|_1^t + 6 \frac{(s - 1)^2}{2} \Big|_1^t + 12 \frac{(s - 1)^3}{3} \Big|_1^t + 8 \frac{(s - 1)^4}{4} \Big|_1^t \\ &= 2s \Big|_1^t + 3(s - 1)^2 \Big|_1^t + 4(s - 1)^3 \Big|_1^t + 2(s - 1)^4 \Big|_1^t \\ &= 2(t - 1) + 3(t - 1)^2 + 4(t - 1)^3 + 2(t - 1)^4 \end{aligned}$$

Solution 2: We have

$$\begin{aligned} \int_1^t \{1 + [1 + 2(s - 1)]^3\} ds &= \left[\begin{array}{l} 1 + 2(s - 1) = u \\ d(1 + 2(s - 1)) = du \\ 2ds = du \\ ds = \frac{1}{2}du \end{array} \right] = \frac{1}{2} \int_{1+2(1-1)}^{1+2(t-1)} (1 + u^3) du \\ &= \frac{1}{2} \int_1^{2t-1} (1 + u^3) du \\ &= \frac{1}{2} \left(u + \frac{u^4}{4} \right) \Big|_1^{2t-1} \\ &= \frac{1}{2} \left[\left(2t - 1 + \frac{(2t - 1)^4}{4} \right) - \left(1 + \frac{1^4}{4} \right) \right] \\ &= \frac{1}{2} \left(2t - \frac{9}{4} + \frac{(2t - 1)^4}{4} \right) \end{aligned}$$

One can check that $\frac{1}{2} \left(2t - \frac{9}{4} + \frac{(2t - 1)^4}{4} \right)$ is identically equal to $2(t - 1) + 3(t - 1)^2 + 4(t - 1)^3 + 2(t - 1)^4$.