5.1 2. $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(\ell) = 0$

① $\lambda = 0$

$y'' = 0 \Rightarrow y = c_1 x + c_2$

$y' = c_1$

$0 = y'(0) = c_1 \Rightarrow c_1 = 0.$

$\Rightarrow y = c_2$ is an eigenfn. w/ eigenvalue $\lambda = 0$.

② $\lambda < 0$. $y = c_1 \cosh\sqrt{-\lambda}x + c_2 \sinh\sqrt{-\lambda}x$

$y' = c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}x + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}x.$

$\begin{cases} y'(0)=0 \\ y'(\ell)=0 \end{cases} \Rightarrow \begin{cases} c_2\sqrt{-\lambda} = 0 \Rightarrow c_2 = 0 \\ c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,\ell = 0 \end{cases} \Rightarrow c_1 = 0$ since $\sinh z > 0$ for $z > 0$.

③ $\lambda > 0$. $y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$

$y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

Boundary condition implies $\begin{cases} c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0 \\ -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\ell = 0 \end{cases} \Rightarrow \sqrt{\lambda}\,\ell = n\pi, n \in \mathbb{Z}.$

$\Rightarrow \lambda = \frac{n^2\pi^2}{\ell^2}.$

$\Rightarrow y_n = c_1 \cos\frac{n\pi}{\ell}x.$ is an eigenfn w/ eigenvalue $\lambda = \frac{n^2\pi^2}{\ell^2}.$

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8. $y'' - 2y' + (1+\lambda)y = 0$. $y(0) = 0$, $y(1) = 0$

$r^2 - 2r + (1+\lambda) = 0.$

$r = \frac{2 \pm \sqrt{4 - 4(1+\lambda)}}{2} = 1 \pm \frac{\sqrt{-4\lambda}}{2}.$

① If $\lambda = 0$. $r = 1$. Then $y = c_1 e^t + c_2 t e^t.$

Boundary conditions $\Rightarrow \begin{cases} c_1 = 0 \\ c_2 \cdot e = 0 \end{cases} \Rightarrow c_2 = 0.$ So $y = 0$.

② $\lambda < 0$. $r = 1 \pm \frac{\sqrt{-4\lambda}}{2}$ is a real #.

$= 1 \pm \sqrt{-\lambda}$

Then $y = c_1 e^{(1+\sqrt{-\lambda})t} + c_2 e^{(1-\sqrt{-\lambda})t}$

Boundary conditions implies $\begin{cases} 0 = y(0) = c_1 + c_2 \\ 0 = y(1) = c_1 e^{1+\sqrt{-\lambda}} + c_2 e^{1-\sqrt{-\lambda}} \end{cases}$

This has a set of nontrivial s/n iff.

$\det \begin{pmatrix} 1 & 1 \\ e^{1+\sqrt{-\lambda}} & e^{1-\sqrt{-\lambda}} \end{pmatrix} \neq 0 \iff e^{1-\sqrt{-\lambda}} - e^{1+\sqrt{-\lambda}} = 0$

$\iff 1-\sqrt{-\lambda} = 1+\sqrt{-\lambda} \Rightarrow \lambda \neq 0$ ↯ We assumed. $(\lambda < 0)$.

③ $\lambda > 0$. $r = 1 \pm \sqrt{-\lambda} = 1 \pm \sqrt{\lambda}\, i$.

Then $y = \cancel{c_1 \cos(1+\sqrt{\lambda})t + c_2 \sin(1+\sqrt{\lambda})t}$ $e^{t}(c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t)$

Boundary condition $\Rightarrow \begin{cases} 0 = y(0) = c_1 \\ 0 = y(1) = e \cdot c_2 \sin\sqrt{\lambda} \end{cases} \Rightarrow \sqrt{\lambda} = n\pi \Rightarrow \lambda = n^2\pi^2$.

So $y_n = e^{t} c_2 \sin n\pi t$ is a nontrivial s/n if $\lambda = n^2\pi^2$ for $n \in \mathbb{Z}$