3.9. 2. $\dot{\vec{x}} = \begin{pmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x}$

$p(\lambda) = (1-\lambda)(-3-\lambda)(1-\lambda) + 5(1-\lambda) = (1-\lambda)(\lambda^2 + 2\lambda + 2)$

Eigenvalues of A are $\lambda_1 = 1$, $\lambda_{2,3} = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

① If $\lambda_1 = 1$, $\begin{pmatrix} 0 & -5 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an e.v. for $\lambda_1 = 1$

② $\lambda_2 = -1 + i$. $\begin{pmatrix} 2-i & -5 & 0 \\ 1 & -2-i & 0 \\ 0 & 0 & 2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \Rightarrow \vec{v}_2 = \begin{pmatrix} 2+i \\ 1 \\ 0 \end{pmatrix}$

is an e.v. for $\lambda_2 = -1 + i$.

(Rmk. how to find $\vec{v}_2$ quickly? You don't have to entirely solve for $\begin{pmatrix} \vdots \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0}$ using augmented matrix as lots of you did in the quiz. The point here is to just find one sln. From the 3rd row eqn $0 \cdot v_1 + 0 \cdot v_2 + (2-i) v_3 = 0$, I get $v_3 = 0$ immediately. r1 eqn & r2 eqn should be essentially the same (just multiple of each other), so I picked r2 eqn $v_1 + (-2-i) v_2 = 0$ as it looks easier. Then I take $v_2 = 1$, which makes $v_1 = 2 + i$.)

Now, $e^{(-1+i)t} \begin{pmatrix} 2+i \\ 1 \\ 0 \end{pmatrix} = e^{-t} \cdot (\cos t + i \sin t) \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$

$\cancel{= e^{-t} (2\cos t - \sin t}$

$$= e^{-t}\left(\cos t\begin{pmatrix}2\\1\\0\end{pmatrix} - \sin t\begin{pmatrix}1\\0\\0\end{pmatrix} + i\cos t\begin{pmatrix}1\\0\\0\end{pmatrix} + i\sin t\begin{pmatrix}2\\1\\0\end{pmatrix}\right)$$

$$= \underbrace{e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\\ 0\end{pmatrix}}_{=:\vec{x}_2(t)} + i\,\underbrace{e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\\ 0\end{pmatrix}}_{=:\vec{x}_3(t)}.$$

Then $\vec{x}(t) = c_1 e^{t}\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\\ 0\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\\ 0\end{pmatrix}$

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6. $\dot{\vec{x}} = \begin{pmatrix}3 & -2\\ 4 & -1\end{pmatrix}\vec{x}$. $\vec{x}(0) = \begin{pmatrix}1\\5\end{pmatrix}$.

$$p(\lambda) = \begin{vmatrix}3-\lambda & -2\\ 4 & -1-\lambda\end{vmatrix} = (\lambda-3)(\lambda+1)+8 = \lambda^2 - 2\lambda + 5$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$ are eigenvalues for A.

For $\lambda = 1+2i$, $\begin{pmatrix}2-2i & -2\\ 4 & -2-2i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = 0 \Rightarrow \vec{v} = \begin{pmatrix}1\\1-i\end{pmatrix}$ is an e.v.

Then $e^{(1+2i)t}\begin{pmatrix}1\\1-i\end{pmatrix} = e^{t}(\cos 2t + i\sin 2t)\left[\begin{pmatrix}1\\1\end{pmatrix} - i\begin{pmatrix}0\\1\end{pmatrix}\right]$

$$= e^{t}\begin{pmatrix}\cos 2t\\ \cancel{\cos 2t\,\sin 2t}\ \cos 2t + \sin 2t\end{pmatrix} + i e^{t}\begin{pmatrix}\sin 2t\\ -\cos 2t + \sin 2t\end{pmatrix}$$

So $\vec{x}(t) = c_1 e^{t}\begin{pmatrix}\cos 2t\\ \cos 2t + \sin 2t\end{pmatrix} + c_2 e^{t}\begin{pmatrix}\sin 2t\\ -\cos 2t + \sin 2t\end{pmatrix}$.

Since $\begin{pmatrix}1\\5\end{pmatrix} = \vec{x}(0)$, $\begin{pmatrix}1\\5\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\-1\end{pmatrix} \Rightarrow \begin{cases}c_1 = 1\\ c_2 = -4\end{cases}$

$$\Rightarrow \vec{x}(t) = -4e^{t}\begin{pmatrix}\sin 2t\\ -\cos 2t + \sin 2t\end{pmatrix} + e^{t}\begin{pmatrix}\cos 2t\\ \cos 2t + \sin 2t\end{pmatrix}$$