3.8. 2. $\dot{\vec{x}} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}$

$0 = P(\lambda) = \begin{vmatrix} -2-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (2+\lambda)(\lambda-3)+4 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1)$

$\Rightarrow \lambda = 2, -1.$

If $\lambda=2$, $\begin{pmatrix} -4 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0} \Rightarrow -4v_1+v_2=0 \Rightarrow v_2 = 4v_1$

$\Rightarrow \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is one eigenvector of $\lambda=2$.

If $\lambda=-1$, $\begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is one eigenvector.

$\Rightarrow \vec{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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6. $\dot{\vec{x}} = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{pmatrix}}_{=:A} \vec{x}$

Note that $A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (x_1+2x_2+3x_3+6x_4)\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}$

So $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}$ is an eigenvector w/ eigenvalue $1+2\cdot3+3\cdot5+6\cdot7=64$

Also $\lambda=0$ is an eigenvalue w/ eigenvectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where $x_1+2x_2+3x_3+6x_4=0$.

Hence $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{6} \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{3} \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$ are 3 l.i. eigenvectors for $\lambda=0$

Therefore, $\vec{x} = c_1 e^{64t}\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{6} \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{3} \end{pmatrix} + c_4\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix}.$

8. $\dot{\vec{x}} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \vec{x}$. $\quad \vec{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

$$0 = p(\lambda) = \begin{vmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$$

$$\Rightarrow \lambda = 4, -1$$

If $\lambda = 4$, $\begin{pmatrix} -3 & -3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector.

If $\lambda = -1$, $\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector.

So $\vec{x} = c_1 e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

But $\vec{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$, $\quad c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$$\Rightarrow \begin{cases} c_1 + 3c_2 = 0 \\ -c_1 + 2c_2 = 5 \end{cases} \Rightarrow \begin{cases} c_1 = -3 \\ c_2 = 1 \end{cases}$$

$$\Rightarrow \vec{x}(t) = -3e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$