5.1. 4. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & -1 \\ 1 & 3 & 10 \end{pmatrix} \vec{x} = \vec{0}$

$\det A = 33 - 2(21) - 3(3) \neq 0$

So it has a unique sln.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 3 & -1 & 0 \\ 1 & 3 & 10 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & +1 & -5 & 0 \\ 0 & 1 & 13 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow \vec{x} = \vec{0}$ is the unique sln.

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5. $\begin{pmatrix} 1 & 2 & 1 \\ -3 & -2 & 1 \\ 6 & 8 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$. $\det A = -12 - 2(-12) + (-12) = 0.$

So it has infinitely many or no solutions.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -3 & -2 & 1 & 5 \\ 6 & 8 & 2 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 5+9=14 \\ 0 & 1 & 1 & \frac{7}{2} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 + x_2 = -\frac{1}{2} \\ x_2 + x_3 = \frac{7}{2} \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{2} - x_2 \\ x_3 = \frac{7}{2} - x_2 \end{cases}$$

$\Rightarrow \vec{x} = \begin{pmatrix} -\frac{1}{2} - x_2 \\ x_2 \\ \frac{7}{2} - x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{7}{2} \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ where x_2 can be any const.

Note. You may notice my answer is different from book's answer $\vec{x}' = c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ \frac{7}{2} \\ 0 \end{pmatrix}$, but indeed they are the same, if you let $c = \frac{7}{2} - x_2$, for any x_2.

10. $\begin{pmatrix} \lambda & 1 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & \lambda \end{pmatrix} \vec{x} = \vec{0}$.

If this system has a nontrivial s/n, it must have ∞-many s/ns, which implies $\det A = 0$.

$0 = \det A = \lambda(\lambda^2 - 9) - (\lambda + 3) + 3(3 + \lambda)$

$= \lambda(\lambda + 3)(\lambda - 3) + 2(\lambda + 3)$

$= (\lambda + 3)(\lambda^2 - 3\lambda + 2)$

$\Rightarrow \lambda = -3, 1, 2.$

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19. $L(x_1, x_2) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$. Take $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2)$

a). $L(c\vec{x}) = L(cx_1, cx_2) = \begin{pmatrix} cx_1 + cx_2 \\ cx_1 - cx_2 \end{pmatrix} = c \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} = c \cdot L(x_1, x_2) = c \cdot L(\vec{x})$.

$L(\vec{x} + \vec{y}) = L(x_1 + y_1, x_2 + y_2) = \begin{pmatrix} x_1 + y_1 + x_2 + y_2 \\ x_1 + y_1 - (x_2 + y_2) \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \end{pmatrix}$

$= L(\vec{x}) + L(\vec{y})$

$\Rightarrow L$ is a linear trans.

b). suppose $\vec{x} = (x_1, x_2) \in S^\circ$, S° is the unit circle.
I want to show that $L(\vec{x})$ is on the circle w/ radius 2.

$L(x_1, x_2) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$.

$(x_1 + x_2)^2 + (x_1 - x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 + x_1^2 - 2x_1x_2 + x_2^2$

$= 2(x_1^2 + x_2^2) = 2.$

So $L(\vec{x})$ is on the circle w/ radius 2 if $\vec{x}$ is on unit circle.