3.4.2. $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix} \vec{x}$ $\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = 2x_1 - x_2 + 2x_3 \end{cases}$

Let $x_1 = y$. $x_2 = y'$, $x_3 = y''$.

Then the above eqns are equivalent to

$y^{(3)} - 2y'' + y' - 2y = 0$

$r^3 - 2r^2 + r - 2 = 0$

$(r-2)(r^2+1) = 0$

$r = 2, \pm i$

Then $y_1 = e^{2t}$, $y_2 = \cos t$ $y_3 = \sin t$ are slns to the 3rd-order hom. eqn.

Then $\vec{x}_1(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} \cos t \\ -\sin t \\ -\cos t \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \end{pmatrix}$.

For any $\vec{x}(t)$ sln to $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix} \vec{x}$,

$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$

Now to check if $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are l.i.

At $t=0$, $\vec{x}_1(0) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

If $c_1 \vec{x}_1(0) + c_2 \vec{x}_2(0) + c_3 \vec{x}_3(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for some c_1, c_2, c_3,

or $\begin{cases} c_1 + c_2 = 0 \\ 2c_1 + c_3 = 0 \\ 4c_1 - c_2 = 0 \end{cases}$ $\Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$ $\Rightarrow \vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are l.i.

So $\{\vec{x}_1(t) \vec{x}_2(t), \vec{x}_3(t)\}$ forms a basis.

6. $\vec{x}' = \begin{pmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{pmatrix} \vec{x}$. $\vec{x}_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ e^{4t} \\ e^{4t} \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} e^{6t} \\ 0 \\ e^{6t} \end{pmatrix}$.

We know the dim for the sln space for this eqn is 3.

Therefore, if I show $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are slns, are they are l.i., then they must span to sln space, and therefore forms a basis.

Now first check if $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are slns.

$LHS = \vec{x}_1' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix}$, $RHS = \begin{pmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix} = LHS$.

So $\vec{x}_1$ is a sln.

Using same method I can check that $\vec{x}_2, \vec{x}_3$ are also slns.

Then I need to show they're l.i.

Take $t=0$. $\vec{x}_1(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. $\vec{x}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{x}_3(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

If for some c_1, c_2, c_3 that

$$c_1 \vec{x}_1(0) + c_2 \vec{x}_2(0) + c_3 \vec{x}_3(0) = \vec{0},$$

$$\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0. \end{cases}$$

So $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are l.i., and therefore forms a basis.

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8. The dim for the sln space should be 3, while there are 2 vectors given. So they do not form a basis.