

3.3.2. $\begin{pmatrix} 1 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Suppose $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases} \Rightarrow c_1 = c_2 = c_3 = 0$$

Thus they are linearly indep.

6 a). $V = \{ \text{polynomials of degree} \leq 2 \}$.

Let $p_0 = 1, p_1 = t, p_2 = t^2$.

Clearly they are l.i.

Also, they span V :

Take any $p \in V$. $p = a_0 + a_1 t + a_2 t^2$.

Then $p = a_0 \cdot p_0 + a_1 \cdot p_1 + a_2 t^2$.

So $\{p_0, p_1, p_2\}$ forms a basis for V .

$\Rightarrow \dim V = 3$

b). $p_1 = (t-1)^2, p_2 = (t-2)^2, p_3 = (t-1)(t-2)$.

Suppose $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$

$$0 = c_1 (t-1)^2 + c_2 (t-2)^2 + c_3 (t-1)(t-2)$$

$$= c_1 (t^2 - 2t + 1) + c_2 (t^2 - 4t + 4) + c_3 (t^2 - 3t + 2)$$

$$= (c_1 + c_2 + c_3)t^2 + (-2c_1 - 4c_2 - 3c_3)t + (c_1 + 4c_2 + 2c_3)$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ -2c_1 - 4c_2 - 3c_3 = 0 \\ c_1 + 4c_2 + 2c_3 = 0 \end{cases} \Rightarrow \begin{cases} 3c_2 + c_3 = 0 \\ -2c_2 - c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

So p_1, p_2, p_3 are l.i.

Since V has $\dim 3$, $\{p_1, p_2, p_3\}$ form a basis

8. [⊙] I will sketch the main ideas to show

$V := \{y : y''' + y = 0 \text{ w/ } y(0) = 0\}$ is a v.s.

V is a subset of the space of all functions.

Also, if $f, g \in V$,

$f + cg$ also satisfies $(f+cg)''' + (f+cg) = 0$,

and $(f+cg)(0) = f(0) + cg(0) = 0$.

Moreover, the zero func. is in V .

so V is a subspace.

ⓐ Find a basis.

I'll solve the eqn $y^{(3)} + y = 0$, $y(0) = 0$
to find a basis.

[Rmk. You can first think about the dimension of this space
For a 3rd-order ^{hom.} diff. eqn., you'll get 3 hom. ~~eq~~ solutions.
But you also have 1 initial condition; this is a
further requirement for your hom. slns. Therefore,
you'll expect the dim of the space is decreased by 1
to 2.]

The characteristic eqn for $y^{(3)} + y = 0$ is

$$r^3 + 1 = 0.$$

One sln

$$r = -1 \quad (y_1 = e^{-t})$$

So $(r+1)$

is one

component of $r^3 + 1$.

You can find the other 2 slns by long division.

$$\begin{array}{r} \Gamma + 1 \mid \overline{\Gamma^3 + 0\Gamma^2 + 0\Gamma + 1} \\ \underline{\Gamma^3 + \Gamma^2} \\ -\Gamma^2 + 0\Gamma + 1 \\ \underline{-\Gamma^2 - \Gamma} \\ \Gamma + 1 \end{array}$$

So $(\Gamma^3 + 1) = (\Gamma + 1)(\Gamma^2 - \Gamma + 1)$.

$$\Gamma^2 - \Gamma + 1 = 0$$

$$\Rightarrow \Gamma = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Then $e^{(\frac{1}{2} + i\frac{\sqrt{3}}{2})t} = e^{\frac{1}{2}t} (\underbrace{\cos \frac{\sqrt{3}}{2}t}_{y_2} + i \underbrace{\sin \frac{\sqrt{3}}{2}t}_{y_3})$

So $y = c_1 \underbrace{e^{-t}}_{y_1} + e^{\frac{1}{2}t} (c_2 \cos \frac{\sqrt{3}}{2}t + c_3 \sin \frac{\sqrt{3}}{2}t)$

This is the general soln.

But we also have $y(0) = 0$

$$\Rightarrow 0 = y(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$$

$$\begin{aligned} \Rightarrow y &= c_1 e^{-t} + e^{\frac{1}{2}t} (-c_1 \cos \frac{\sqrt{3}}{2}t + c_3 \sin \frac{\sqrt{3}}{2}t) \\ &= c_1 (e^{-t} - e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t) + c_3 \cdot e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t \end{aligned}$$

So if $f_1 = e^{-t} - e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t$, $f_2 = e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$, $\{f_1, f_2\}$ forms a basis.

(I also need to check f_1, f_2 are lin. ind., but I'll skip this part here)