3.3.2. $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$.

Suppose $c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases} \Rightarrow \quad c_1 = c_2 = c_3 = 0$$

Thus they are linearly indep.

6 a). $V = \{$ polynomials w/ degree $\leq 2 \}$.

Let $p_0 = 1$, $p_1 = t$, $p_2 = t^2$.

Clearly they are l. i.

Also, they span V:

Take any $p \in V$. $\quad p = a_0 + a_1 t + a_2 t^2$.

Then $p = a_0 \cdot p_0 + a_1 \cdot p_1 + a_2 t^2$.

So $\{p_0, p_1, p_2\}$ forms a basis for V.

$\Rightarrow \dim V = 3$

b). $p_1 = (t-1)^2$, $p_2 = (t-2)^2$, $p_3 = (t-1)(t-2)$.

Suppose $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$

$$\begin{aligned} 0 &= c_1(t-1)^2 + c_2(t-2)^2 + c_3(t-1)(t-2) \\ &= c_1(t^2 - 2t + 1) + c_2(t^2 - 4t + 4) + c_3(t^2 - 3t + 2) \\ &= (c_1 + c_2 + c_3)t^2 + (-2c_1 - 4c_2 - 3c_3)t + (c_1 + 4c_2 + 2c_3) \end{aligned}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ -2c_1 - 4c_2 - 3c_3 = 0 \\ c_1 + 4c_2 + 2c_3 = 0 \end{cases} \Rightarrow \begin{cases} 3c_2 + c_3 = 0 \\ -2c_2 - c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

So p_1, p_2, p_3 are l. i.

Since V has dim 3, $\{p_1, p_2, p_2\}$ forms a basis

8. ① I will sketch the main ideas to show

$V := \{ y : y''' + y = 0 \text{ w/ } y(0) = 0 \}$ is a v.s.

V is a subset of the space of all functions.

Also, if $f, g \in V$,

$f + cg$ also satisfies $(f+cg)''' + (f+cg) = 0$,

and $(f+cg)(0) = f(0) + cg(0) = 0$.

Moreover, the zero func. is in V.

So V is a subspace.

② Find a basis.

I'll solve the eqn $y^{(3)} + y = 0$, $y(0) = 0$

to find a basis.

[Rmk: You can first think about the dimension of this space

For a 3rd-order hom. diff. eqn., you'll get 3 hom. solutions.

But you also have 1 ininitial condition; this is a further requirement for your hom. slns. Therefore, you'll expect the dim of the space is decreased by 1 to 2.]

The characteristic eqn for $y^{(3)} + y = 0$ is

$r^3 + 1 = 0$. One sln is $r = -1$. $(y_1 = e^{-t})$

So $(r+1)$ is one component of $r^3 + 1$.

You can find the other 2 slns by long division.

$$\begin{array}{r|l} r+1 & r^3 + 0r^2 + 0r + 1 \\ & \underline{r^3 + r^2} \\ & -r^2 + 0r + 1 \\ & \underline{-r^2 - r} \\ & r + 1 \end{array}$$

So $(r^3+1) = (r+1)(r^2-r+1)$.

$r^{\cancel{3}2} - r + 1 = 0$

$\Rightarrow r = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Then $e^{(\frac{1}{2}+i\frac{\sqrt{3}}{2})t} = e^{\frac{1}{2}t}(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t)$

$$\underbrace{e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t}_{y_2} \qquad \underbrace{e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}_{y_3}$$

So $y = c_1 \underbrace{e^{-t}}_{y_1} + e^{\frac{1}{2}t}(c_2\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t)$

This is the general s/n.

But we also have $y(0) = 0$

$\Rightarrow 0 = y(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$

$\Rightarrow y = c_1 e^{-t} + e^{\frac{1}{2}t}(-c_1\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t)$

$= c_1(e^{-t} - e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t) + c_3 \cdot e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$.

So if $f_1 = e^{-t} - e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$, $f_2 = e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$,

$\{f_1, f_2\}$ forms a basis.

(I also need to check f_1, f_2 are lin. ind.,
but I'll skip this part here)