3.3

#9. $V = \{ p(t) = a_0 + a_1 t + a_2 t^2 : \ p(0) + 2p'(0) + 3p''(0) = 0 \}$

- First we show that V is a vector space by showing it's a subspace of the space of all real-valued functions:

① if $p(t) = 0$ then $p(0) + 2p'(0) + 3p''(0) = 0 \Rightarrow 0 \in V$.

② if $p(t), q(t) \in V$, then $p(0) + 2p'(0) + 3p''(0) = 0$ and $q(0) + 2q'(0) + 3q''(0) = 0$

for $p+q$, we have

$$(p+q)(0) + 2(p+q)'(0) + 3(p+q)''(0)$$
$$= p(0) + q(0) + 2p'(0) + 2q'(0) + 3(p''(0) + q''(0))$$
$$= 0.$$

Thus V is closed under addition.

③ if $p(t) \in V$, then $p(0) + 2p'(0) + 3p''(0) = 0$

For any constant $c \in \mathbb{R}$, $(cp)(0) + 2(cp)'(0) + 3(cp)''(0)$

$$= c\,p(0) + 2c\,p'(0) + 3c\,p''(0)$$
$$= c\,(p(0) + 2p'(0) + 3p''(0)) = 0$$

Thus V is closed under scalar multiplication.

Therefore V is a subspace of the space of all functions. V is a vector space.

(You can also 'translate' the condition $p(0) + 2p'(0) + 3p''(0) = 0$ into $a_0 + 2a_1 + 6a_2 = 0$, as we did in the next part, and show that all polynomials $a_0 + a_1 t + a_2 t^2$ with $a_0 + 2a_1 + 6a_2 = 0$ form a vector space. This is equivalent.)

- Next we find a basis for V.

First, by plugging in $p(t) = a_0 + a_1 t + a_2 t^2$ to $p(0) + 2p'(0) + 3p''(0) = 0$,

we get: $p(0) = a_0$, $p'(t) = a_1 + 2a_2 t$, $p'(0) = a_1$,

$p''(t) = 2a_2$, $p''(0) = 2a_2$

$\Rightarrow p(0) + 2p'(0) + 3p''(0) = 0$ is equivalent to $a_0 + 2a_1 + 6a_2 = 0$.

Thus the vector space

$$V = \{p(t) = a_0 + a_1 t + a_2 t^2 : \quad a_0 + 2a_1 + 6a_2 = 0\}$$
$$= \{a_0 + a_1 t + a_2 t^2 : \quad a_0 = -2a_1 - 6a_2\},$$

since we notice that $a_0 + 2a_1 + 6a_2 = 0$ restricts the coefficients such that one of them can be completely determined by the others.

Thus $$V = \{a_0 + a_1 t + a_2 t^2 : \quad a_0 = -2a_1 - 6a_2\}$$
$$= \{(-2a_1 - 6a_2) + a_1 t + a_2 t^2\}$$
$$= \{a_1(-2+t) + a_2(t^2 - 6)\}$$

Thus any $p(t) \in V$ can be represented as
$$p(t) = a_1(t-2) + a_2(t^2-6),$$
or to say, $p(t)$ is a linear combination of $(t-2)$, (t^2-6).
Since $(t-2)$ and (t^2-6) are linearly independent, (you can verify this by $c_1(t-2) + c_2(t^2-6) = 0$ or by observing that they are not scalar multiples of each other), they form a basis for V.
Thus V is a vector space with dimension 2, and a set of basis for V is $\{t^2-6, \ t-2\}$.

* We can also write $a_1 = -\frac{1}{2}a_0 - 3a_2$ or $a_2 = -\frac{1}{6}a_0 - \frac{1}{3}a_1$
and $p(t) = a_0 + a_1 t + a_2 t^2 = a_0 + a_1 t + (-\frac{1}{6}a_0 - \frac{1}{3}a_1)t^2$
$$= a_0(1 - \frac{1}{6}t^2) + a_1(t - \frac{1}{3}t^2)$$
then $p(t)$ is a linear combination of $(1-\frac{1}{6}t^2)$, $(t - \frac{t^2}{3})$
Thus $\{1 - \frac{t^2}{6}, \ t - \frac{t^2}{3}\}$ is also a set of basis for V.

* You can also observe that V has dimension 2 (since one of the coefficients is determined by the other 2).
then find two arbitrary polynomials in V such that they are linearly independent. For example, since $a_0 + 2a_1 + 6a_2 = 0$

try $a_0 = 1$, $a_1 = -\frac{1}{2}$, $\xRightarrow{a_2 = 0}$ $p_1(t) = 1 - \frac{1}{2}t$.

$a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{6}$ $\Rightarrow$ $p_2(t) = 1 - \frac{1}{6}t^2$.

Since p_1, p_2 are linearly independent and $\dim V = 2$, they form a basis of V.

* You must notice that $p(0) + 2p'(0) + 3p''(0) = 0$ is <u>not</u> a differential equation. It is <u>not</u> the same as $y + 2y' + 3y'' = 0$, since that means $y(t) + 2y'(t) + 3y''(t) = 0$ for all t, not only for $t = 0$. The vector space V is <u>not</u> the space of solutions to the differential equation $y + 2y' + 3y'' = 0$.