3.2. 2. $\{ \vec{x} : \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ w/ } x_1 + x_2 + x_3 = 0 \}$.

Let A be the set defined above.

A is a subset of $\underbrace{\{ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \}}_{=\mathbb{R}^3}$, where the latter is a v.s.

So it suffices to show if A is closed under addition & scalar.

Take $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in A$.

Then $x_1 + x_2 + x_3 = 0$ & $y_1 + y_2 + y_3 = 0$.

I want to check if $\vec{x} + \vec{y} \in A$.

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}.$$

Since $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0$,

$\vec{x} + \vec{y} \in A$.

I also want to see if $c \cdot \vec{x} \in A$ for any scalar c.

$$c \cdot \vec{x} = c \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix},$$ and $cx_1 + cx_2 + cx_3 = c(x_1 + x_2 + x_3) = 0$.

So $c\vec{x} \in A$. (Trivially $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in A$.)

Therefore, A is closed under + & $\cdot$, thus A is a subspace of $\mathbb{R}^3$, and is a v.s. by itself.

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3. $B := \{ \vec{x} : x_1^2 + x_2^2 + x_3^2 = 1 \}$.

Clearly $\vec{0} \notin B$.

So B is not a v.s.

6. $H := \{ \vec{x} \cancel{\mathbb{R}^3} : x_1 + x_2 + x_3 = 0,\ x - x_2 + 2x_3 = 0,\ 3x - x_2 + 5x_3 = 0 \}$.

Again, it suffices to show if H is a subspace of $\mathbb{R}^3$.

· Trivially, $\vec{0} \in H$.

· Take $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in H$.

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}.$$

$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = 0$ since $\begin{cases} x_1 + x_2 + x_3 = 0 \\ y_1 + y_2 + y_3 = 0 \end{cases}$.

$(x_1 + y_1) - (x_2 + y_2) + 2(x_3 + y_3) = 0$ since $\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ y_1 - y_2 + 2y_3 = 0 \end{cases}$

and $3(x_1 + y_1) - (x_2 + y_2) + 5(x_3 + y_3) = 0$ since $\begin{cases} 3x_1 - x_2 + 5x_3 = 0 \\ 3y_1 - y_2 + 5y_3 = 0 \end{cases}$.

Therefore $\vec{x} + \vec{y} \in H$.

· Take any $c \in \mathbb{R}$.

$$c\vec{x} = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix}.$$

$cx_1 + cx_2 + cx_3 = 0$

$cx_1 - cx_2 + 2 \cdot (cx_3) = 0$ b/c $\vec{x} \in H$.

$3(cx_1) - cx_2 + 5c(cx_3) = 0$

$\Rightarrow c\vec{x} \in H$.

So H is a subspace of $\mathbb{R}^3$.

8. $G := \{ f : f \text{ is differentiable} \}$.

We know that the space of all func.'s (call it F) forms a v.s., and G is a subset of that.

Then it suffices to show G as a subspace of F.

- The zero element here is the zero function (the func. that sends everything to 0) Note here we're in the space of functions. All the elements are func.'s, not numbers. So you need to consider the zero element as a func., not a number.

  Clearly the zero func. is differentiable.

- Take $f, g \in G$. ~~Then~~

  $f+g$ is also diff. and its derivative $(f+g)' = f'+g'$.

  f', g' exist because f, g are diff.

- For any $c \in \mathbb{R}$, $c \cdot f$ is also diff., and $(cf)' = c \cdot f'$.

Thus, G is a ~~sup~~ subspace of F.

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10 $K := \{$ ~~f : f is diff~~ $y : y'' + y = \cos t \}$.

It is not a v.s. since the zero function is not in K.

$0 = 0'' + 0 \neq \cos t$, as the LHS is constantly 0, while the RHS $= \cos t$ is a func. that changes w/ t.