3.10. 2. $\dot{\vec{x}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \vec{x}$.

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)\big((1-\lambda)(4-\lambda)+2\big) - \big(2(4-\lambda)-3\big) + 4+3(1-\lambda)$$

$$= (1-\lambda)(\lambda^2-5\lambda+6) - 8+2\lambda+3+4+3-3\lambda$$

$$= -\lambda^3+6\lambda^2-11\lambda+6+\cancel{6}2-\lambda$$

$$= -\lambda^3+6\lambda^2-12\lambda+8$$

$$= -(\lambda-2)^3.$$

So $\lambda = 2$ is the eigenvalue w/ multiplicity 3.

$$(A-2I)\vec{v} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases} -v_1+v_2+v_3=0 \\ 2v_1-v_2-v_3=0 \\ -3v_1+2v_2+2v_3=0 \end{cases} \Rightarrow \begin{cases} +v_1=0 \\ v_2=-v_3 \\ \cancel{2v_2+v_3=0} \end{cases} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \text{ is an e.v.}$$

Hence $\vec{x}_1(t) = e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is one sln.

Now consider sln's to $\vec{0} = (A-2I)^2\vec{v} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\vec{v}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow -v_1+v_2+v_3=0$$

$\Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is a sln and $(A-2I)\vec{v}_2 \neq 0$.

So $\vec{x}_2(t) = e^{2t}\big(I+t(A-2I)\big)\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = e^{2t}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} 2 \\ 1+2t \\ 1-2t \end{pmatrix}$

Finally, consider slns to $\vec{0} = (A-2I)^3\vec{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is one sln and $(A-\lambda I)^2\vec{v}_3 \neq 0$.

Then $\vec{x}_3(t) = e^{2t}\left(I + t(A-2I) + \frac{t^2}{2}(A-2I)^2\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$= e^{2t}\left(\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$$

$$= e^{2t}\begin{pmatrix} t \\ -t+\frac{t^2}{2} \\ 1+2t-\frac{t^2}{2}\end{pmatrix}$$

Therefore, $\vec{x}(t) = e^{2t}\left(c_1 \begin{pmatrix}0\\1\\-1\end{pmatrix} + c_2\begin{pmatrix}2\\1+2t\\1-2t\end{pmatrix} + c_3\begin{pmatrix}t\\-t+\frac{t^2}{2}\\1+2t-\frac{t^2}{2}\end{pmatrix}\right)$