3.1

2. $\dfrac{d^3y}{dt^3} + \cos y = e^t$

Set $x_1 = y$, $x_2 = \dfrac{dy}{dt}$, $x_3 = \dfrac{d^2y}{dt^2}$. Then $\dfrac{d^3y}{dt^3} = \dfrac{dx_3}{dt}$, and

$$\begin{cases} x_2 = \dfrac{dx_1}{dt} \\ x_3 = \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} = e^t - \cos x_1 \end{cases}$$

4. $\dfrac{d^2y}{dt^2} + 3\dfrac{dz}{dt} + 2y = 0$ & $\dfrac{d^2z}{dt^2} + 3\dfrac{dy}{dt} + 2z = 0$.

Set $x_1 = y$, $x_2 = y'$, $x_3 = z$, $x_4 = z'$.

Then $\dfrac{d^2y}{dt^2} = \dfrac{dx_2}{dt}$, $\dfrac{d^2z}{dt^2} = \dfrac{dx_4}{dt}$. I have

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_3}{dt} = x_4 \\ \dfrac{dx_2}{dt} = -3x_4 - 2x_1 & (\to \text{from first eqn.}) \\ \dfrac{dx_4}{dt} = -3x_2 - 2x_3 & (\to \text{from second eqn.}) \end{cases}$$

8. $$\begin{cases} \dot{x}_1 = x_1 + x_2 - x_3 \\ \dot{x}_2 = 3x_1 - x_2 + 4x_3 \\ \dot{x}_3 = -x_1 - x_2 \end{cases} \qquad \begin{cases} x_1(0) = 0 \\ x_2(0) = -1 \\ x_3(0) = -1 \end{cases}$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & -1 & 0 \end{pmatrix}$

Then $\dot{\vec{x}} = A\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

11. $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ -1 & -1 & 2 \end{pmatrix}.$

$$A\vec{x} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - x_3 \\ 3x_1 + 4x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix}$$

a). $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

b) $\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

c) $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$

14. $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$

We want to compute $A\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$.

First write $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Suppose $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$

Then $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix} + \begin{pmatrix} b \\ -b \\ -b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \\ a-b \end{pmatrix} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$

So $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Then $$A\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = A\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right) = A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2A\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 6 \\ 12 \end{pmatrix}$$