Section 3.1

#5. (a) $\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$

Then $\vec{x}'(t) = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y-y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}(t).$

(b) $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $\vec{x}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}(t)$

Then $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$= \begin{pmatrix} 0 \cdot x_1 + 1 \cdot x_2 \\ (-1) \cdot x_1 + (-1) \cdot x_2 \end{pmatrix}$

$= \begin{pmatrix} x_2 \\ -x_1 - x_2 \end{pmatrix}$

$\Rightarrow \quad \frac{dx_1}{dt} = x_2$

$\frac{d^2x_1}{dt^2} = \frac{dx_2}{dt} = -x_1 - x_2 = -x_1 - \frac{dx_1}{dt}$

$\Rightarrow \quad \frac{d^2x_1}{dt^2} + \frac{dx_1}{dt} + x_1 = 0$

$\Rightarrow y = x_1$ satisfies the equation $y'' + y' + y = 0$. $\square$

#15. Know that $A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$. $A\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-3\\-6\end{pmatrix}$.

From Exercise 12, $Ae_i = A\begin{pmatrix}0\\ \vdots\\1\\ \vdots\\0\end{pmatrix}\leftarrow i\text{-th}$ is the <u>i-th column</u> of A

So $V_1 = A\begin{pmatrix}1\\0\end{pmatrix}$ is the 1st column of A,

$V_2 = A\begin{pmatrix}0\\1\end{pmatrix}$ is the 2nd column of A.

In order to find the matrix A, it's enough for us to find V_1 and V_2.

Since $\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}$, by the linearity of matrix-vector multiplication,

$$A\begin{pmatrix}1\\1\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix}+A\begin{pmatrix}0\\1\end{pmatrix} = V_1+V_2$$

$$\Rightarrow \quad V_1+V_2 = A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}. \qquad (1)$$

Similarly, $\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}$

$$A\begin{pmatrix}1\\-1\end{pmatrix} = A\left(\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\right) = A\begin{pmatrix}1\\0\end{pmatrix}-A\begin{pmatrix}0\\1\end{pmatrix} = V_1-V_2$$

$$\Rightarrow \quad V_1-V_2 = A\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix} \qquad (2)$$

From equations (1). (2) we get

$$(1)+(2) \Rightarrow V_1 = \frac{1}{2}\left[\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}3\\6\end{pmatrix}\right]=\begin{pmatrix}\frac{7}{2}\\4\end{pmatrix}$$

$$(1)-(2) \Rightarrow V_2 = \frac{1}{2}\left[\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}3\\6\end{pmatrix}\right]=\begin{pmatrix}\frac{1}{2}\\-2\end{pmatrix}$$

$$\Rightarrow \quad A=[V_1, V_2] = \begin{bmatrix}\frac{7}{2} & \frac{1}{2}\\ 4 & -2\end{bmatrix}.$$

(You can verify that $A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$, $A\begin{pmatrix}+1\\-1\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix}$) $\square$