2.8 P197

2. $\overbrace{y''-ty}^{L[y]}=0$

Set $y=\sum_{n=0}^{\infty} a_n t^n$

$y' = \sum_{n=0}^{\infty} n\cdot a_n t^{n-1}$

$y'' = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}$

Then
$$
\begin{aligned}
L[y] &= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - t\cdot\sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{0}^{\infty} a_n t^n \\
&\overset{\substack{j=n-2\\ n=j+2}}{=} \sum_{0}^{\infty}(j+2)(j+1)a_{j+2}t^j - \underbrace{\sum_{n=0}^{\infty} a_n t^{n+1}}_{= a_0 t + \sum_{n=1}^{\infty} a_n t^{n+1} \overset{j=n+1}{=} a_0 t + \sum_{j=2}^{\infty} a_{j-1}t^j} \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \left(a_0 t + \sum_{n=2}^{\infty} a_{n-1}t^n\right) \\
&= 2\cdot a_2 t^0 + 6a_3 t - a_0 t + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=2}^{\infty} a_{n-1}t^n \\
&= 2a_2 + 6a_3 t - a_0 t + \sum_{n=2}^{\infty}\left((n+2)(n+1)a_{n+2} - a_{n-1}\right)t^n \\
&= 0
\end{aligned}
$$

$$
\Rightarrow \begin{cases} 2a_2 = 0 \\ 6a_3 - a_0 = 0 \\ (n+2)(n+1)a_{n+2} - a_{n-1} = 0 \end{cases}
\Rightarrow \begin{cases} a_2 = 0 \\ a_3 = \dfrac{a_0}{6} \\ a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)} \end{cases}
$$

(1) $a_0 = 1$, $a_1 = 0$

Then $a_3 = \frac{1}{6}$, $a_4 = \dfrac{a_1}{4\cdot 3} = 0$, $a_5 = \dfrac{a_2}{5\cdot 4} = 0$,

$a_6 = \dfrac{a_3}{6\cdot 5} = \frac{1}{6}\cdot\frac{1}{30}$.

So $a_k = 0$ if k is not $3n$ for $n\in\mathbb{N}$.

$$a_{3n} = \frac{a_{3(n-1)}}{3n\cdot(3n-1)} = \frac{1}{3n(3n-1)}\cdot\frac{a_{3(n-2)}}{(3n-3)(3n-4)}\cdots\frac{a_3}{6\cdot 5}$$

So $y_1 = \sum_{n=0}^{\infty} a_n t^n = 1+\frac{1}{6}t^3+\sum_{n=1}^{\infty} a_{3n}t^{3n}$, where a_{3n} is defined above.

② $a_0 = 0, \quad a_1 = 1.$

Then $a_{3k} = 0, \quad a_{3k+2} = 0$

$$a_4 = \frac{a_1}{4\cdot 3} = \frac{1}{12}, \qquad a_7 = \frac{a_4}{7\cdot 6} = \frac{1}{7\cdot 6}\cdot\frac{1}{4\cdot 3}$$

$$a_{3k+1} = \frac{a_{3k-2}}{(3k+1)(3k)} = \frac{1}{(3k+1)(3k)}\cdot\frac{a_{3k-5}}{(3k-2)(3k-3)}\cdot \cdots \cdot \frac{a_1}{4\cdot 3}$$

Then, $y_2 = \sum_{n=0}^{\infty} a_n t^n = t + \sum_{n=1}^{\infty} a_{3n+1}\cdot t^{3n+1}$, where a_{3n+1} is defined above

6. $\overbrace{y'' + t^2 y}^{L[y]} = 0. \quad y(0) = 2, \quad y'(0) = -1$

Set $y = \sum_{n=0}^{\infty} a_n t^n.$

$$y'' = \sum_{n=0}^{\infty} n(n-1)\, a_n t^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

$$L[y] = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+2} \qquad \sum_{n=1}^{\infty} a_{4n}\cdot 4n\cdot t^{4n-1}$$

$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2} t^n + \sum_{n=2}^{\infty} a_{n-2} t^n$$

$$= 2a_2 + 6a_3 t + \sum_{n=2}^{\infty}\left[(n+2)(n+1)\, a_{n+2} + a_{n-2}\right] t^n$$

$$= 0$$

$$\Rightarrow \begin{cases} 2a_2 = 0 \\ 6a_3 = 0 \\ (n+2)(n+1)\, a_{n+2} + a_{n-2} = 0 \end{cases} \qquad \Rightarrow \begin{cases} a_2 = 0 \\ a_3 = 0 \\ a_{n+2} = -\dfrac{a_{n-2}}{(n+2)(n+1)} \end{cases}$$

① $a_0 = 1, \quad a_1 = 0.$

Then $a_{4k+1} = a_{4k+2} = a_{4k+3} = 0.$

$$a_4 = -\frac{a_0}{4\cdot 3} = -\frac{1}{12}, \qquad a_8 = -\frac{a_4}{8\cdot 7} = -\frac{1}{8\cdot 7}\left(-\frac{1}{4\cdot 3}\right)$$

$$a_{4(k+1)} = -\frac{a_{4k}}{(4k+4)(4k+3)} = (-1)^{k+1}\left(\frac{1}{(4k+4)(4k+3)}\cdot\frac{1}{(4k)(4k-1)}\cdot \cdots \cdot \frac{1}{4\cdot 3}\right)$$

By initial conditions, we set $a_0 = 2$, $a_1 = -1$.

Then $a_{4k+2} = a_{4k+3} = 0$.

$$a_4 = -\frac{a_0}{4\cdot 3} = -\frac{2}{4\cdot 3}, \quad a_8 = -\frac{a_4}{8\cdot 7} = \frac{2}{4 3\cdot 8\cdot 7}$$

$$a_{4k} = -\frac{a_{4k-4}}{(4k)\cdot(4k-1)} = -\frac{1}{(4k)(4k-1)} \cdot \frac{-a_{4k-8}}{(4k-4)(4k-5)} \cdots \frac{-2}{4\cdot 3}$$

$$= (-1)^{k} \frac{2}{4^k \cdot k! \cdot (4k-1)(4k-5)\cdots 3}$$

$$a_5 = \frac{-a_1}{5\cdot 4} = \frac{1}{5\cdot 4}, \quad a_9 = \frac{-a_5}{9\cdot 8} = -\frac{1}{9\cdot 8}\cdot\frac{1}{5\cdot 4}$$

$\cancel{a_{4k+1}}$

$$a_{4k+1} = \frac{-a_{4k-3}}{(4k+1)(4k)} = \frac{-1}{(4k+1)(4k)} \cdots \frac{-a_1}{5\cdot 4}$$

$$= (-1)^{k+1} \frac{1}{4^k k! (4k+1)(4k-3)\cdots(5)}$$

So $y = \sum_{n=0}^{\infty} a_n t^n = 2 + \sum_{n=1}^{\infty} a_{4n} t^{4n} + (-1)t + \sum_{n=1}^{\infty} a_{4n+1} t^{4n+1}$,

where a_{4n}, a_{4n+1} are defined above.

P203. 2. $\overbrace{2t^2y'' + 3ty' - y}^{L[y]} = 0.$

Set $y = t^r$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

Then $L[y] = 2t^2 \cdot r(r-1)t^{r-2} + 3trt^{r-1} - t^r$

$= 2r(r-1)t^r + r \cdot 3t^r - t^r$

$= t^r(2r(r-1) + 3r - 1) = 0$

$\Rightarrow 2r^2 + r - 1 = 0$

$r = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1 \text{ or } \frac{1}{2}$

So $y = c_1 t^{-1} + c_2 t^{1/2}$.

Rmk. You can also go directly into the characteristic eqn. However, in order to use that, you need to make sure the coeff. of t^2y'' is 1. So here you'll first divide by 2 to get $t^2y'' + \frac{3}{2}ty' - \frac{1}{2}y = 0$.
From here you have $r^2 + (\frac{3}{2} - 1)r + (-\frac{1}{2}) = 0$.

6. $(t-2)^2y'' + 5(t-2)y' + 4y = 0.$

Set $y = (t-2)^r$. $y' = r(t-2)^{r-1}$. $y'' = r(r-1)(t-2)^{r-2}$.

Then I have $0 = r(r-1)(t-2)^r + 5r(t-2)^r + 4(t-2)^r$

$= (t-2)^r(r^2 + 4r + 4)$

$\Rightarrow r = -2$ w/ multiplicity 2.

So $y = (c_1 + c_2 \ln t)(t-2)^{-2}$