Section 2.8

#3. $\underbrace{(2+t^2)y'' - ty' - 3y}_{L[y]} = 0.$

Set $y = \sum_{n=0}^{\infty} a_n t^n$ then $y' = \sum_{n=0}^{\infty} n a_n t^{n-1}$, $y'' = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}$

$$L[y] = (2+t^2)y'' - ty' - 3y$$

$$= (2+t^2)\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=0}^{\infty} n a_n t^{n-1} - 3\sum_{n=0}^{\infty} a_n t^n$$

let $m = n-2$ $(n = m+2)$

$$= \sum_{n=0}^{\infty} 2n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n t^n - \sum_{n=0}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} 3a_n t^n$$

$$= \sum_{m=-2}^{\infty} 2(m+2)(m+1)a_{m+2} t^m + \sum_{n=0}^{\infty} [(n(n-1) - n - 3)a_n] t^n$$

(when $m = -2, -1$, the terms are zero)

$$= \sum_{m=0}^{\infty} 2(m+2)(m+1)a_{m+2} t^m + \sum_{n=0}^{\infty} (n^2 - 2n - 3)a_n t^n$$

$$= \sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} + (n-3)(n+1)a_n] t^n$$

$$= 0$$

$$\Rightarrow \quad 2(n+2)(n+1)a_{n+2} + (n-3)(n+1)a_n = 0.$$

$$\Rightarrow \quad \boxed{a_{n+2} = -\frac{n-3}{2(n+2)} a_n} \qquad (*).$$

① $a_0 = 1$, $a_1 = 0$.

Then from $(*)$, $a_3 = a_5 = \cdots = a_{2n+1} = 0$. $\forall n = 1, 2, \ldots$

Only the even coefficients are nonzero.

From $a_{n+2} = -\dfrac{n-3}{2(n+2)} a_n$ we get:

$$a_0 = 1$$

$$a_2 = -\frac{(0-3)}{2(0+2)} a_0 = (-1) \times \frac{(-3)}{2\times 2} = \frac{3}{4}$$

$$a_4 = -\frac{(2-3)}{2(2+2)} a_2 = (-1) \times \frac{(-1)}{2\times 4} \times \frac{3}{4} = \frac{3}{32}.$$

$$\vdots$$

$$a_{2n} = -\frac{(2n-2-3)}{2(2n-2+2)} a_{2n-2}$$

$$= (-1)\frac{(2n-5)}{2\cdot(2n)} a_{2n-2}$$

$$= (-1)\frac{(2n-5)}{2\cdot(2n)} \cdot \cancel{(-1)} \frac{(2n-7)}{2\cdot(2n-2)} a_{2n-4}$$

$a_2 = (-1)\cdot\frac{(-3)}{2\times 2}$ $(n=1)$

$$= (-1)\frac{(2n-5)}{2\cdot(2n)} \cdot (-1) \cdot \frac{(2n-7)}{2\cdot(2n-2)} \times \cdots \times (-1) \cdot \frac{(-1)}{2\times 4} \times \boxed{(-1)\cdot\frac{(-3)}{2\times 2}} \times 1$$

$$= (-1)^n \cdot \frac{(-3)\times(-1)\times 1\times 3\times \cdots \times (2n-7)\times(2n-5)}{2^n\cdot(2n)\times(2n-2)\times(2n-4)\times\cdots\times 4\times 2}$$

$$= (-1)^n \cdot \frac{(-3)\times(-1)\times 1\times 3\times\cdots\times(2n-5)}{2^n\cdot 2^n\cdot(n(n-1)(n-2)\cdots\times 2\times 1)}$$

$$= (-1)^n \cdot \frac{(-3)\times(-1)\times 1\times 3\times\cdots\times(2n-5)}{2^{2n}\cdot n!}$$

$\Rightarrow y_0(t) = \sum_{n=0}^{\infty} a_{2n} t^{2n}$ where $a_{2n} = (-1)^n \dfrac{(-3)\times(-1)\times 1\times 3\times\cdots\times(2n-5)}{2^{2n}\cdot n!}$

② let $a_0 = 0$, $a_1 = 1$.

Then $a_{2n} = 0$.

$$a_1 = 1$$

$$a_3 = -\frac{(1-3)}{2\times(1+2)} a_1 = -\frac{(-2)}{6}\times 1 = \frac{1}{3}.$$

$$a_5 = -\frac{(3-3)}{2\times(3+2)} a_3 = 0$$

$$a_7 = a_9 = \cdots = 0$$

$$\Rightarrow y_1(t) = a_1 t + a_3 t^3 + 0$$

$$= t + \frac{1}{3}t^3.$$

Therefore the general solution $y(t) = a_0 y_0(t) + a_1 y_1(t)$

$$= a_0\left(\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-3)\times(-1)\times\cdots\times(2n-5)}{2^{2n}\cdot n!} t^{2n}\right) + a_1\left(t + \frac{1}{3}t^3\right)$$

$\square$

#5. $t(2-t)y'' - 6(t-1)y' - 4y = 0$, $y(1) = 1$, $y'(1) = 0$

Since the initial condition is given at $t = 1$,

write the coefficients of the equation as polynomials in $(t-1)$:

$$t(2-t) = 2t - t^2 = -(t^2 - 2t)$$

$$= -(t^2 - 2t + 1 - 1) = -(t-1)^2 + 1$$

$$\Rightarrow (-(t-1)^2 + 1)\, y'' - 6(t-1)y' - 4y = 0$$

$$\Rightarrow ((t-1)^2 - 1)\, y'' + 6(t-1)y' + 4y = 0 \qquad (*)$$

let $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$.

$$y'(t) = \sum_{n=0}^{\infty} n a_n (t-1)^{n-1}$$

$$y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n (t-1)^{n-2}.$$

$$L[y] = ((t-1)^2 - 1) y'' + 6(t-1) y' + 4y$$

$$= \sum_{n=0}^{\infty} n(n-1) a_n (t-1)^n - \sum_{n=0}^{\infty} n(n-1) a_n (t-1)^{n-2}$$

$$+ \cancel{6} \sum_{n=0}^{\infty} 6n a_n (t-1)^n + \sum_{n=0}^{\infty} 4 a_n (t-1)^n$$

$$= \sum_{n=0}^{\infty} (n(n-1) a_n + 6 n a_n + 4 a_n)(t-1)^n$$

$$- \sum_{m=-2}^{\infty} (m+2)(m+1) a_{m+2} (t-1)^m$$

$$= \sum_{n=0}^{\infty} (n^2 + 5n + 4) a_n (t-1)^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (t-1)^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)(n+4) a_n - (n+2)(n+1) a_{n+2} \right] (t-1)^n = 0$$

$$\Rightarrow \quad (n+1)(n+4) a_n - (n+2)(n+1) a_{n+2} = 0$$

$$\Rightarrow \quad \boxed{a_{n+2} = \frac{n+4}{n+2} a_n}$$

Notice that we are given the initial condition $y(1) = 1$, $y'(1) = 0$,
and that $y(1) = a_0$, $y'(1) = a_1$

$\Rightarrow$ we only need to consider $a_0 = 1$, $a_1 = 0$

Thus $a_0 = 1$, $a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = a_{2n+1} = 0$.

$$a_2 = \frac{0+4}{0+2} a_0 = \frac{4}{2} \times 1 = 2$$

$$a_4 = \frac{2+4}{2+2} a_2 = \frac{6}{4} \times \frac{4}{2} = \frac{6}{2} = 3.$$

$$a_6 = \frac{4+4}{4+2} a_4 = \frac{8}{6} \times \frac{6}{4} \times \frac{4}{2} = \frac{8}{2} = 4.$$

$\vdots$

$$a_{2n} = \frac{(2n-2)+4}{(2n-2)+2} a_{2n-2} = \frac{2n+2}{\cancel{2n}} \times \frac{\cancel{2n}}{2n-2} \times \cdots \times \frac{\cancel{6}}{\cancel{4}} \times \frac{\cancel{4}}{2}$$

$$= \frac{2n+2}{2} = n+1.$$

(or: you can directly find the pattern from $a_2 = 2$, $a_4 = 3$, $a_6 = 4$, ...)

$$\Rightarrow \quad y(t) = \sum_{n=0}^{\infty} a_{2n} (t-1)^{2n} = \sum_{n=0}^{\infty} (n+1)(t-1)^{2n}.$$

is the solution to equation with initial conditions $y(1) = 1$, $y'(1) = 0$.

$\square$