2.5. 2. $y''+4y+4y=te^{\alpha t}$

$r^2+4r+4=0$. $r=-2$ w/ multiplicity 2.

①. If $\alpha \neq -2$:

$\psi = (A_0+A_1t)e^{\alpha t}$

$\psi' = e^{\alpha t}(A_1+\alpha(A_0+A_1t))$

$\psi'' = e^{\alpha t}(\alpha A_1+\alpha A_1+\alpha^2(A_0+A_1t))$

Plug ψ into the diff. eqn:

$\psi''+4\psi'+4\psi = te^{\alpha t}$

$\|$

$e^{\alpha t}(\underline{2\alpha A_1}+\underline{\alpha^2A_0}+\alpha^2A_1t+\underline{4A_1}+\underline{4\alpha A_0}+4\alpha A_1t+\underline{4A_0}+A_1t)$

$\|$

$e^{\alpha t}((2\alpha+4)A_1+(\alpha^2+4\alpha+4)A_0+(\alpha^2A_1+4\alpha A_1+A_1)t)$

$$\Rightarrow\begin{cases}(2\alpha+4)A_1+(\alpha+2)^2A_0=0\\ \alpha^2A_1+4\alpha A_1+A_1=1\end{cases}\Rightarrow\begin{cases}A_0=-\dfrac{2\alpha+4}{(\alpha^2+4\alpha+1)(\alpha+2)^2}=\dfrac{-2}{(\alpha^2+4\alpha+1)(\alpha+2)}\\ A_1=\dfrac{1}{\alpha^2+4\alpha+1}\end{cases}$$

So $\psi=(A_0+A_1t)e^{\alpha t}$, w/ A_0, A_1 calculated above.

② If $\alpha=-2$:

$\psi = t^2(A_0+A_1t)e^{-2t}$

$\psi' = 2t(A_0+A_1t)e^{-2t}+t^2e^{-2t}(A_1-2A_0-2A_1t)$

$= e^{-2t}(2tA_0+2t^2A_1+t^2A_1-2t^2A_0-2t^3A_1)$

$= e^{-2t}(2A_0t+(3A_1-2A_0)t^2-2A_1t^3)$

$\psi'' = e^{-2t}(2A_0+(6A_1-4A_0)t-6A_1t^2-4A_0t+(-6A_1+4A_0)t^2+4A_1t^3)$

$= e^{2t}(2A_0+(6A_1-8A_0)t+(4A_0-12A_1)t^2+4A_1t^3)$

Plug in ψ: $e^{-2t}(2A_0+(6A_1-\cancel{8A_0})t+(\cancel{4A_0}-\cancel{12A_1})t^2+\cancel{4A_1}t^3+\cancel{8A_0}t+(12A_1-8A_0)t^2$
$-\cancel{8A_1}t^3+\cancel{4A_0}t^2+\cancel{4A_1}t^3)=t\cdot e^{-2t}$

$$\Rightarrow \begin{cases} 2A_0 = 0 \\ 6A_1 \cancel{- 8A_0 + 8A_0} = 1 \end{cases} \Rightarrow \begin{array}{l} A_0 = 0 \\ A_1 = \frac{1}{6}. \end{array}$$

Then $\psi = t^2 (\frac{1}{6} t) e^{-2t}$.

---

7. $y'' + 4y = t\sin 2t$.

Consider $y'' + 4y = t \cdot e^{2it}$ $(*)$.

The imaginary sln to $(*)$ will be the sln to the orignial eqn.

$r^2 + 4 = 0 \rightarrow r = \pm 2i$.

So $\alpha = 2i$ is one sln to the characteristic eqn.

Let $\psi = t(A_0 + A_1 t) e^{2it}$

$$\psi' = e^{2it}(A_0 + A_1 t + t \cdot A_1 + 2it(A_0 + A_1 t))$$

$$= e^{2it}(A_0 + (2A_1 + 2iA_0)t + 2iA_1 t^2)$$

$$\psi'' = e^{2it}(2A_1 + 2iA_0 + 4iA_1 t + 2iA_0 + (4iA_1 - 4A_0)t - 4A_1 t^2)$$

$$= e^{2it}(2A_1 + 4iA_0 + (8iA_1 - 4A_0)t - 4A_1 t^2)$$

Plug ψ into $(*)$:

$$e^{2it}(2A_1 + 4iA_0 + (8iA_1 - 4A_0)t - 4A_1 t^2 + \cancel{4(A_0 + A_1} \; 4A_0 t + 4A_1 t^2)$$

$$= t \cdot e^{2it}$$

$$\Rightarrow \begin{cases} 2A_1 + 4iA_0 = 0 \\ 8iA_1 - 4A_0 + 4A_0 = 1 \end{cases} \Rightarrow \begin{cases} A_0 = \frac{1}{16} \\ A_1 = \frac{1}{8i} = \frac{-i}{8} \end{cases}$$

So $\psi = (\frac{1}{16} + \frac{-i}{8}t)t \cdot e^{2it} = (\frac{1}{16} - \frac{i}{8}t)t(\cos 2t + i\sin 2t)$

$= (\text{Real part}) + i\,\underbrace{t(\frac{1}{16}\sin 2t - \frac{t}{8}\cos 2t)}$ what we want.

10. $y'' - 2y' + 5y = 2(\cos^2 t)e^t$.

We don't know how to solve anything w/ $\cos^2 t$ on the RHS.
However, we can use double angle formula to change $\cos^2 t$ into $\frac{\cos 2t + 1}{2}$.

Then the eqn becomes $y'' - 2y' + 5y = \cancel{2} \cdot \frac{\cos 2t + 1}{\cancel{2}} \cdot e^t$

$= \cos 2t \cdot e^t + e^t$.

If y_1 solves ① $y'' - 2y' + 5y = \cos 2t \cdot e^t$, and
y_2 solves ② $y'' - 2y' + 5y = e^t$, then
$y = y_1 + y_2$ solves the eqn.

①. Consider $y'' - 2y' + 5y = e^{2it} \cdot e^t = e^{\underset{\alpha}{\underbrace{(1+2i)}}t}$

$r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$. so α is one sln to chara. eqn.

Let $y = A_0 t \cdot e^{(1+2i)t}$

$y' = A_0 e^{(1+2i)t}(1 + (1+2i)t)$

$y'' = A_0 e^{(1+2i)t}(1+2i + (1+2i) + (1+2i)^2 t)$

Plug in y to ①: $e^{(1+2i)t}(2(1+2i)A_0 + (1+2i)^2 A_0 t - 2A_0 - 2A_0(1+2i)t$
$+ 5A_0 t) = e^{(1+2i)t}$

$\Rightarrow \begin{cases} 2(1+2i)A_0 - 2A_0 = 1 \\ (1+2i)^2 A_0 - 2A_0(1+2i) + 5A_0 = 0 \end{cases} \Rightarrow A_0 = \frac{1}{4i} = \frac{-i}{4}$

Then $y = \frac{-i}{4} t e^{(1+2it)} = \frac{-i}{4} t e^t(\cos(2t) + i\sin 2t) = t \cdot e^t \cdot \frac{1}{4}\sin 2t + (Im.)$

So $y_1 = \frac{1}{4}\sin 2t \, t e^t$

② $\alpha_2 = 1$ is not a sln to charac. eqn.

So $y_2 = B_0 e^t$. $y_2' = B_0 e^t$. $y_2'' = B_0 e^t$

Plug in y_2 to ②: $y_2'' - 2y_2' + 5y_2 = (B_0 - 2B_0 + 5B_0)e^t = e^t$

$\Rightarrow B_0 = \frac{1}{4}$

So $y_2 = \frac{1}{4} e^t$.

So $y = y_1 + y_2 = \frac{1}{4} \sin 2t \cdot t e^t + \frac{1}{4} e^t$.

Hint for 15, 16: use formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}$

So $\cos t \cdot \cos 2t = \frac{1}{2}(\cos 3t + \cos t)$ (here $\alpha = 3$, $\beta = 1$)