Section 2.5

#11. $y'' + y' - 6y = \sin t + te^{2t}$ (*)

Let ψ_1 be a solution of: $y'' + y' - 6y = \sin t$ ①

ψ_2 be a solution of: $y'' + y' - 6y = te^{2t}$ ②

Then (by linearity of the operator $L[y] = y'' + y' - 6y$),

$\psi = \psi_1 + \psi_2$ is a solution of equation (*)

- First we find a particular solution of ①:

$$y'' + y' - 6y = \sin t = Im[e^{it}]$$

Consider $y'' + y' - 6y = e^{it}$ (1')

Characteristic equation $r^2 + r - 6 = 0 \Rightarrow r = -3$ or $r = 2$.

$\alpha = i$ is not a root of $r^2 + r - 6 = 0$, therefore

$$\psi = A_0 e^{\alpha t} = A_0 e^{it}, \quad \psi' = i A_0 e^{it}, \quad \psi'' = i^2 A_0 e^{it}$$

$$\psi'' + \psi' - 6\psi = e^{it}$$

$$\Rightarrow A_0 (i^2 + i - 6) e^{it} = e^{it}$$

$$\Rightarrow A_0 (-1 + i - 6) = 1$$

$$\Rightarrow A_0 = \frac{1}{-7+i} = \frac{-7-i}{(-7+i)(-7-i)} = \frac{-7-i}{50}.$$

$\Rightarrow \psi = \frac{-7-i}{50} e^{it}$ is a solution of (1').

Therefore $Im[\psi] = Im\left[\frac{-7-i}{50}(\cos t + i \sin t)\right]$

$$= Im\left[-\frac{7}{50}\cos t - \frac{7}{50} i \sin t - \frac{1}{50} i \cos t - \frac{i}{50} \cdot i \sin t\right]$$

$$= -\frac{1}{50}\cos t - \frac{7}{50}\sin t.$$

$\psi_1 = Im[\psi] = -\frac{1}{50}\cos t - \frac{7}{50}\sin t$ is a particular solution of ①.

· Next we find a particular solution of ②:

$$y'' + y' - 6y = t e^{2t}.$$

Since $\alpha = 2$ is a single root of $r^2 + r - 6 = 0$,

$$\psi = te^{2t}(A_0 + A_1 t) = e^{2t}(A_0 t + A_1 t^2)$$

(simplify first, then take derivative)

$$\psi' = 2e^{2t}(A_0 t + A_1 t^2) + e^{2t}(A_0 + 2A_1 t)$$

$$= e^{2t}(2A_0 t + 2A_1 t^2 + A_0 + 2A_1 t)$$

$$= e^{2t}(A_0 + 2(A_1 + A_0)t + 2A_1 t^2)$$

$$\psi'' = 2e^{2t}(A_0 + 2(A_1 + A_0)t + 2A_1 t^2) + e^{2t}(2(A_1 + A_0) + 4A_1 t)$$

$$= e^{2t}(4A_0 + 2A_1 + (4A_0 + 8A_1)t + 4A_1 t^2)$$

$$\psi'' + \psi' - 6\psi = te^{2t}$$

$$\Rightarrow e^{2t}\left[4A_0 + 2A_1 + (4A_0 + 8A_1)t + 4A_1 t^2 + A_0 + 2(A_1 + A_0)t + 2A_1 t^2 - 6(A_0 t + A_1 t^2)\right] = te^{2t}$$

$$\Rightarrow e^{2t}(5A_0 + 2A_1 + 10A_1 t) = te^{2t}$$

$$\Rightarrow (5A_0 + 2A_1) + 10A_1 t = t$$

$$\Rightarrow \begin{cases} 5A_0 + 2A_1 = 0 \\ 10A_1 = 1 \end{cases} \Rightarrow \begin{aligned} A_0 &= -\frac{1}{25} \\ A_1 &= \frac{1}{10} \end{aligned}$$

Thus $\psi_2 = te^{2t}\left(-\frac{1}{25} + \frac{1}{10}t\right)$ is a particular solution to ②

$\psi = \psi_1 + \psi_2 = -\frac{1}{50}(\cos t + 7\sin t) + te^{2t}\left(-\frac{1}{50} + \frac{1}{10}t\right)$ is a particular solution of (*)