2.4. 2. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = \underbrace{te^{2t}}_{g(t)}$

For the hom. eqn, the characteristic eqn is

$$r^2 - 4r + 4 = 0 \quad \Rightarrow \quad r = 2 \text{ w/ multiplicity } 2.$$

So $y_1 = e^{2t}$, $y_2 = te^{2t}$. $y_1' = 2e^{2t}$, $y_2' = e^{2t}(2t+1)$

For the particular s/n, $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1'(t) = -\frac{g(t)\,y_2}{W}, \quad u_2' = \frac{g \cdot y_1}{W}.$$

$$W = y_1 y_2' - y_2 y_1' = e^{2t}\cdot e^{2t}(2t+1) - te^{2t}\cdot 2e^{2t} = e^{4t}.$$

Then $u_1' = -\frac{t\cdot e^{2t}\cdot te^{2t}}{e^{4t}} = -t^2$, $u_2' = \frac{te^{2t}\cdot e^{2t}}{e^{4t}} = t$.

$$\Rightarrow u_1(t) = \int -t^2\,dt = -\frac{t^3}{3}, \quad u_2(t) = \int t\,dt = \frac{t^2}{2}$$

So $y_p = y_1 u_1 + y_2 u_2 = e^{2t}\cdot\frac{-t^3}{3} + te^{2t}\cdot\frac{t^2}{2} = \frac{1}{6}t^3 e^{2t}$

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{6}t^3 e^{2t}$$

$$= e^{2t}\left(c_1 + c_2 t + \frac{1}{6}t^3\right)$$

4. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = \underbrace{te^{3t} + 1}_{g(t)}$.

First solve slns for hom. eqn.

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0.$$

$$r = 2,\ 1$$

Then $y_1 = e^{2t}$, $y_2 = e^t$. $y_1' = 2e^{2t}$, $y_2' = e^t$

$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = e^{2t} \cdot e^t - e^t \cdot 2e^{2t} = -e^{3t}$.

Now calculate y_P.

$$u_1' = -\frac{g(t)y_2}{W} = -\frac{(te^{3t}+1)e^t}{-e^{3t}} = te^t + e^{-2t}$$

$$u_2' = \frac{g(t)y_1}{W} = \frac{(te^{3t}+1)e^{2t}}{-e^{3t}} = -te^{2t} - e^{-t}$$

$$\Rightarrow u_1 = \int (te^{t} + e^{-2t})\,dt = \frac{e^{-2t}}{-2} + \underbrace{\int te^t}_{\text{Int. by parts}}$$

$$\begin{pmatrix} u = t & u' = 1 \\ v = e^t & v' = e^t \end{pmatrix} te^t - \int e^t = (t-1)e^t$$

$$= \frac{e^{-2t}}{-2} + (t-1)e^t$$

$$u_2 = -\int (te^{2t} + e^{-t})dt = e^{-t} - \underbrace{\int te^{2t}dt}_{\overset{\text{I.B.P.}}{=} t\cdot\frac{e^{2t}}{2} - \int \frac{e^{2t}}{2}} = e^{-t} - \frac{1}{2}te^{2t} + \frac{e^{2t}}{4}$$

Then $y_P = e^{2t}\left(\frac{e^{-2t}}{-2} + (t-1)e^t\right) + e^t\left(e^{-t} + e^{2t}\left(\frac{1}{4} - \frac{t}{2}\right)\right)$

$$= -\frac{1}{2} + (t-1)e^{3t} + 1 + e^{3t}\left(\frac{1}{4} - \frac{t}{2}\right)$$

$$= \frac{1}{2} + e^{3t}\left(\frac{t}{2} - \frac{3}{4}\right).$$

$$y = c_1 e^{2t} + c_2 e^t + \frac{1}{2} + e^{3t}\left(\frac{t}{2} - \frac{3}{4}\right).$$

2.4. 6. $y'' + 4y' + 4y = \underbrace{t^{5/2}e^{-2t}}_{g(t)}$. $\quad y(0) = y'(0) = 0$.

$r^2 + 4r + 4 = 0. \quad (r+2)^2 = 0. \quad r = -2$

Then $y_1 = e^{-2t}, \quad y_2 = t \cdot e^{-2t}$.

$y_1' = -2e^{-2t}, \quad y_2' = e^{-2t}(-2t+1)$

$W = e^{-2t} \cdot e^{-2t}(-2t+1) - te^{-2t} \cdot (-2e^{-2t}) = e^{-4t}$

$u_1' = -\frac{g(t)\, y_2}{W} = -\frac{t^{5/2}e^{-2t} \cdot te^{-2t}}{e^{-4t}} = -t^{7/2}$.

$\Rightarrow u_1 = \int -t^{7/2}\, dt = -\frac{2t^{9/2}}{9}$

$u_2' = \frac{g(t)\, y_1}{W} = \frac{t^{5/2}e^{-2t} \cdot e^{-2t}}{e^{-4t}} = t^{5/2}$

$\Rightarrow u_2 = \int t^{5/2}\, dt = \frac{2t^{7/2}}{7}$

Hence $\quad y_p = y_1 u_2 + y_2 u_2 = e^{-2t} \cdot \frac{-2t^{9/2}}{9} + te^{-2t} \cdot \frac{2t^{7/2}}{7}$

$= \frac{4}{63} t^{9/2} \cdot e^{-2t}$

$y = c_1 \cdot e^{-2t} + c_2 te^{-2t} + \frac{4}{63} t^{9/2} e^{-2t} = e^{-2t}(c_1 + c_2 t + \frac{4}{63} t^{9/2})$

$y' = e^{-2t}(c_2 + \frac{2}{7} t^{7/2}) + (c_1 + c_2 t + \frac{4}{63} t^{9/2}) \cdot (-2)e^{-2t}$.

$\begin{cases} 0 = y(0) = c_1 \\ 0 = y'(0) = c_2 \end{cases} \Rightarrow y = \frac{4}{63} t^{9/2} e^{-2t}$.