2.1. 3. $L[y](t) = \int_a^t s^2 y(s)\,ds$

In order to show L is linear, I want to show

$L[cy_1 + y_2] = cL[y_1] + L[y_2]$.

$$L[cy_1 + y_2](t) = \int_a^t s^2 (c y_1(s) + y_2(s))\,ds$$

$$= \int_a^t s^2 c y_1(s) + s^2 y_2(s)$$

$$= c\int_a^t s^2 y_1(s)\,ds + \int_a^t s^2 y_2(s)\,ds$$

$$= c \cdot L[y_1](t) + L[y_2](t).$$

Note: The ~~lineari~~ ~~linearity~~ linearity of L is basically coming from the linearity of the integration.

4. $L[y](t) = y'' + p(t)y'(t) + q(t)y(t)$.

We know $L[t^2] = t+1$ & $L[t] = 2t+2$.

If $y(t) = t - 2t^2$ solves $y'' + p(t)y' + q(t)y = 0$,

$L[t-2t^2] = 0$.

$L[t-2t^2] = L[t] - 2L[t^2] = 2t+2 - 2(t+1) = 0$.

Hence $t-2t^2$ is a solution of $y'' + p(t)y' + q(t)y = 0$.

5a). Simply pluging $y_1(t) = \sqrt{t}$, $y_2 = \frac{1}{t}$ into $2t^2y'' + 3ty' - y = 0$ to see if the equality holds.

b). $W = y_1y_2' - y_2y_1' = \sqrt{t}\cdot\left(\frac{-1}{t^2}\right) - \frac{1}{t}\left(\frac{1}{2\sqrt{t}}\right) = \frac{-1}{t^{3/2}} - \frac{1}{2t^{3/2}} = -\frac{3}{2t^{3/2}}$

c) $W[y_1, y_2] = -\frac{3}{2t^{3/2}} \neq 0$ for $0 < t < \infty$.

So y_1, y_2 form a fundamental set of sol.

d). $y = c_1y_1 + c_2y_2 = c_1\sqrt{t} + c_2\frac{1}{t}$. Since $y(1) = 2$, $y'(1) = 1$, I have

$\begin{cases} 2 = c_1 + c_2 \\ 1 = c_1 \ldots \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ \ldots \end{cases} \Rightarrow y = 2\sqrt{t}\ldots$