Section 2.1

6. $$y'' + ty' + y = 0 \qquad (*)$$

(a) $$y_1(t) = e^{-\frac{t^2}{2}}.$$

We show that y_1 satisfies equation $(*)$:

$$y_1' = (-t)e^{-\frac{t^2}{2}}$$

$$y_1'' = (-1)\, e^{-\frac{t^2}{2}} + (-t)\left(-te^{-\frac{t^2}{2}}\right) = (t^2-1)e^{-\frac{t^2}{2}}$$

Then $$y_1'' + ty_1' + y_1 = (t^2-1)\, e^{-\frac{t^2}{2}} + t\cdot(-t)e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}$$
$$= (t^2-1-t^2+1)\, e^{-\frac{t^2}{2}}$$
$$= 0.$$ ①

Next we show that $y_2(t) = e^{-\frac{t^2}{2}} \int_0^t e^{+\frac{s^2}{2}} ds$ satisfies equation $(*)$.

Oberve that $\boxed{y_2(t) = y_1(t) \cdot F(t)}$ ② where $F(t) = \int_0^t e^{+\frac{s^2}{2}} ds$.

From Fundamental Theorem of Calculus,

$$F'(t) = e^{\frac{t^2}{2}}.$$

Therefore, $$y_2' = y_1' \cdot F + y_1 F' = y_1' F + e^{-\frac{t^2}{2}} \cdot e^{\frac{t^2}{2}} = y_1' F + 1$$ ③

$$y_2'' = (y_1'F + 1)'$$
$$= y_1'' F + y_1' F' + 0$$
$$= y_1'' F + \left(-te^{-\frac{t^2}{2}}\right) \cdot e^{\frac{t^2}{2}}$$
$$= y_1'' F - t.$$ ④

Thus $$y_2'' + ty_2' + y_2 \overset{\text{From ②③④}}{=\!=} (y_1''F - t) + t\cdot(y_1'F+1) + y_1 F$$
$$= F\cdot(y_1'' + ty_1' + y_1) - t + t = 0. \quad (\text{Since } y_1'' + ty_1' + y_1 = 0)$$

(b) $W[y_1, y_2](t) = y_1 y_2' - y_2 y_1'$

$= y_1 (y_1' F + 1) - (y_1 F) \cdot y_1'$ (From ② ③)

$= \cancel{y_1 y_1' F} + y_1 - \cancel{y_1 y_1' F}$

$= y_1(t)$

$= e^{-\frac{t^2}{2}}$

(c) Since $W[y_1, y_2](t) = e^{-\frac{t^2}{2}} \neq 0$ (for all t), y_1 and y_2 form a fundamental set of solutions of (*) on $(-\infty, \infty)$.

(d) $y'' + ty' + y = 0$, $y(0) = 0$, $y'(0) = 1$. (**)

$y(t) = C_1 y_1(t) + C_2 y_2(t)$.

Then $y'(t) = C_1 y_1'(t) + C_2 y_2'(t)$.

Plug in $t = 0$, $y_1(0) = 1$, $y_1'(0) = 0$,

$y_2(0) = y_1(0) F(0) = 0$, $y_2'(0) = y_1'(0) F(0) + 1 = 1$.

$$\Rightarrow \begin{cases} 0 = C_1 \cdot 1 + C_2 \cdot 0 \\ 1 = C_1 \cdot 0 + C_2 \cdot 1 \end{cases}$$

$\Rightarrow$ $C_1 = 0$, $C_2 = 1$.

$\Rightarrow$ $y(t) = y_2(t) = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds$ is the solution to the initial value problem (**).