Section 1.9

6. $\underbrace{\frac{y^2}{2} - 2ye^t}_{M} + \underbrace{(y-e^t)}_{N}\frac{dy}{dt} = 0$

$\frac{\partial M}{\partial y} = y - 2e^t$. $\frac{\partial N}{\partial t} = -e^t$

So $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, the equation is not exact.

But since $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} = \frac{y-2e^t-(-e^t)}{y-e^t} = 1$, (can think of it as a function of t.)

we can multiply by $\mu(t)$ on both sides of the equation and make it exact:

$$\mu M + \mu N\frac{dy}{dt} = 0$$

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N)$$

$$\mu\frac{\partial M}{\partial y} = \mu'(t)N + \mu\frac{\partial N}{\partial t}$$

$$\Rightarrow \quad \mu'(t) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N}\mu(t)$$

$$\frac{1}{\mu}\frac{d\mu}{dt} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} = 1$$

$$\Rightarrow \quad \frac{1}{\mu}\frac{d\mu}{dt} = 1$$

$$\ln|\mu| = \int 1\,dt = t + C$$

Take $\mu = e^t$ (it is enough to take one arbitrary μ that satisfies $\mu'(t) = \mu(t)$)

Then the equation becomes

$$\mu M + \mu N \frac{dy}{dt} = 0$$

$$e^t\left(\frac{y^2}{2} - 2ye^t\right) + e^t(y - e^t)\frac{dy}{dt} = 0$$

$$\underbrace{\frac{y^2}{2}e^t - 2ye^{2t}}_{\frac{\partial\phi}{\partial t}} + \underbrace{(e^t y - e^{2t})}_{\frac{\partial\phi}{\partial y}}\frac{dy}{dt} = 0$$

$\Rightarrow$ there exists $\phi(t,y)$ such that

$$\frac{\partial\phi}{\partial t} = \frac{y^2}{2}e^t - 2y e^{2t} \Rightarrow \phi = \frac{y^2}{2}e^t - ye^{2t} + h(y) \quad ①$$

$$\frac{\partial\phi}{\partial y} = e^t y - e^{2t} \Rightarrow \phi = \frac{y^2}{2}e^t - ye^{2t} + k(t) \quad ②$$

① = ②

$$\Rightarrow \frac{y^2}{2}e^t - ye^{2t} + h(y) = \frac{y^2}{2}e^t - ye^{2t} + k(t)$$

$$\Rightarrow \quad h(y) = k(t) = \text{const}$$

Can take both of them to be zero

$$\Rightarrow \quad \phi(t,y) = \frac{y^2}{2}e^t - ye^{2t} + 0 = C.$$

$\square$

7. $\underbrace{2ty^3}_{M} + \underbrace{3t^2y^2}_{N}\frac{dy}{dt} = 0$, $y(1) = 1$.

$\frac{\partial M}{\partial y} = 6ty^2$, $\frac{\partial N}{\partial t} = 6ty^2$.

So the equation is exact.

$M = \frac{\partial \phi}{\partial t} = 2ty^3 \Rightarrow \phi = t^2y^3 + h(y)$

$N = \frac{\partial \phi}{\partial y} = 3t^2y^2 \Rightarrow \phi = t^2y^3 + k(t)$

$\Rightarrow h(y) = k(t) = 0$.

$\phi(t,y) = t^2y^3 = C$

$y = (Ct^{-2})^{\frac{1}{3}}$, if $t \neq 0$

plug in $y(1) = 1$: $1 = y(1) = (C \cdot 1^{-2})^{\frac{1}{3}} \Rightarrow C = 1$.

$\Rightarrow y = t^{-\frac{2}{3}}$ $(t \neq 0)$.

$\square$

9. $\underbrace{3t^2 + 4ty}_{M} + \underbrace{(2y+2t^2)}_{N}\frac{dy}{dt} = 0$, $y(0) = 1$.

$\frac{\partial M}{\partial y} = 4t$, $\frac{\partial N}{\partial t} = 4t$.

So the equation is exact.

$M = \frac{\partial \phi}{\partial t} = 3t^2 + 4ty \Rightarrow \phi = t^3 + 2t^2y + h(y)$

$N = \frac{\partial \phi}{\partial y} = 2y + 2t^2 \Rightarrow \phi = y^2 + 2t^2y + k(t)$

$\Rightarrow h(y) = y^2$, $k(t) = t^3$.

$\Rightarrow \phi(t,y) = t^3 + 2t^2y + y^2 = C$.

plug in $y(0) = 1$: $0^3 + 2 0^2 1 + 1^2 = C \Rightarrow C = 1$.

$\Rightarrow t^3 + 2t^2y + y^2 = 1$.

10. $$\underbrace{y\cos(2t)e^{ty} - 2\sin(2t)e^{ty} + 2t}_{M} + (\underbrace{t\cos(2t)e^{ty} - 3}_{N})\frac{dy}{dt} = 0$$

$y(0) = 0$.

$$\frac{\partial M}{\partial y} = \cos(2t)e^{ty} + y\cos(2t)\cdot t\, e^{ty} - 2\sin(2t)\cdot t\, e^{ty}$$

$$\frac{\partial N}{\partial t} = \cos(2t)e^{ty} + t\cdot(-2\sin(2t))e^{ty} + t\cos(2t)\cdot y\, e^{ty}$$

$$= \frac{\partial M}{\partial y}$$

So it is an exact equation.

$$\phi = \int M\,dt + h(y) = \int N\,dy + k(t)$$

Since integrating $\int M(t,y)\,dt$ is much harder than $\int N(t,y)\,dy$, we use the 2nd way:

$$\phi(t,y) = \int N(t,y)\,dy + k(t)$$

$$= \int (t\cos(2t)e^{ty} - 3)\,dy + k(t)$$

$$= t\cos(2t)\cdot\frac{1}{t}e^{ty} - 3y + k(t)$$

$$= \cos(2t)e^{ty} - 3y + k(t)$$

Since $\frac{\partial \phi}{\partial t} = M$.

$$-2\sin(2t)e^{ty} + \cos(2t)\cdot y\, e^{ty} + k'(t) = M = y\cos(2t)e^{ty} - 2\sin(2t)e^{ty} + 2t$$

$$\Rightarrow k'(t) = 2t$$

$$\Rightarrow k(t) = t^2.$$

$$\Rightarrow \phi(t,y) = \cos(2t)e^{ty} - 3y + t^2 = C$$

Plug in $t=0$, $y=0$: $C=1$