1.4.1. $(1+t^2)\frac{dy}{dt} = 1+y^2$.

$$\frac{dy}{1+y^2} = \frac{dt}{(1+t^2)}$$

Note $\int \frac{1}{1+x^2}\,dx = \arctan x$

So $\arctan y = \arctan t + C$

$y = \tan(\arctan t + C)$

Hint $\frac{\tan(\arctan t) + \tan C}{1 - \tan(\arctan t)\tan C}$

$= \frac{t+C}{1-tC}$

6. $t^2(1+y^2) + 2y\frac{dy}{dt} = 0$. $y(0) = 1$

$2y\frac{dy}{dt} = -t^2(1+y^2)$

$\frac{2y}{1+y^2}\,dy = -t^2\,dt$

$\int \frac{2y}{1+y^2}\,dy = -\int t^2\,dt = \frac{-t^3}{3} + C$

$u = 1+y^2$ // $du = 2y\,dy$

$\int \frac{du}{u} = \ln(1+y^2)$

$1+y^2 = Ce^{-\frac{t^3}{3}}$

$y = \sqrt{Ce^{-\frac{t^3}{3}} - 1}$

(here I take y positive b/c $y(0)=1>0$).

$1 = y(0) = \sqrt{C-1} \Rightarrow C = 2$

4. $\frac{dy}{dt} = e^{t+y+3}$

$= e^{t+3} \cdot e^{y}$

$\frac{dy}{e^y} = dt \cdot e^{t+3}$

$-e^{-y} = e^{t+3} + C$

$e^{-y} = -(e^{t+3} + C)$

$-y = \ln(-e^{t+3} + C)$

$y = -\ln(-e^{t+3} + C)$

9. $\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)}$, $y(0) = -1$.

$\int dy\, 2(y-1) = \int dt\,(3t^2+4t+2)$

$y^2 - 2y = t^3 + 2t^2 + 2t + C$

$y(0) = -1 \Rightarrow$

$1 + 2 = C \Rightarrow C = 3$

So $y^2 - 2y = t^3 + 2t^2 + 2t + 3$

$(y-1)^2 = t^3 + 2t^2 + 2t + 4$

$y = -\sqrt{t^3+2t^2+2t+4} + 1$

(the neg. sign comes from $y(0) = -1$)

1.4.11. $\frac{dy}{dt} = k(a-y)(b-y)$. $y(0)=0$. $a, b>0$.

$\cancel{\frac{dy}{a-y} = k\,dt}$ $\quad \frac{dy}{(a-y)(b-y)} = k\,dt$.

① If $a=b$.

$$\int \frac{dy}{(a-y)^2} = \int k\,dt$$

$$\frac{1}{a-y} = kt + c$$

$$a-y = \frac{1}{kt+c}$$

$$y = a - \frac{1}{kt+c}$$

$y(0) = 0 \Rightarrow 0 = a - \frac{1}{c}$

$\Rightarrow c = \frac{1}{a}$

So $y = a - \frac{1}{kt+\frac{1}{a}}$

$$= \frac{a^2kt}{akt+1}$$

② If $a \neq b$.

$$\int \frac{dy}{(a-y)(b-y)} = \int k\,dt = kt + c$$

$$\frac{1}{(a-y)(b-y)} = \frac{A}{a-y} + \frac{B}{b-y} = \frac{Ab-Ay+Ba-By}{(a-y)(b-y)}$$

$\Rightarrow \begin{cases} Ab+Ba=1 \\ -A-B=0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{b-a} \\ B = \frac{1}{a-b} \end{cases}$

So $\int \frac{dy}{(a-y)(b-y)} = \int \frac{A}{a-y} + \int \frac{B}{b-y}$

$$= \frac{1}{b-a}\int \frac{1}{a-y} + \frac{1}{a-b}\int \frac{1}{b-y}$$

$$= \frac{-1}{b-a}\ln|a-y| - \frac{1}{a-b}\ln|b-y|$$

$$= \frac{1}{a-b}(\ln|a-y| - \ln|b-y|)$$

$$= \frac{1}{a-b}\ln\left|\frac{a-y}{b-y}\right| = kt+c$$

$$\ln\left|\frac{a-y}{b-y}\right| = (a-b)(kt) + c$$

$$\frac{a-y}{b-y} = c \cdot e^{(a-b)kt}$$

$y(0)=0 \Rightarrow 0 = \frac{a}{b} = c$.

So $\frac{a-y}{b-y} = \frac{a}{b} \cdot e^{(a-b)kt}$

$$a-y = (b)\cdot\frac{a}{b}\cdot e^{(a-b)kt} - y \cdot \frac{a}{b} e^{(a-b)kt}$$

$$y\left(\frac{a}{b}\cdot e^{(a-b)kt} - 1\right) = a\left(e^{(a-b)kt} - 1\right)$$

$$y = \frac{a\left(e^{(a-b)kt}-1\right)}{\frac{a}{b}e^{(a-b)kt}-1}$$