1. $y' = 2t(y+1)$. $\overset{y_0}{\overset{\shortparallel}{y(0)}} = 0$.

$y(t) = y_0 + \int_0^t 2s(y(s)+1)\,ds$

$y_{n+1}(t) = y_0 + \int_0^t 2s(y_n(s)+1)\,ds$

Hence. $y_1 = 0 + \int_0^t 2s(0+1)\,ds$

$= \int_0^t 2s\,ds = t^2$

$y_2 = \int_0^t 2s(s^2+1)\,ds = \frac{t^4}{2} + t^2$

$\vdots$

$y_n = \int_0^t 2s(y_{n-1}+1) = \int_0^t 2s\left(1+s^2+\frac{s^4}{2}+\cdots+\frac{s^{2(n-1)}}{(n-1)!}\right)ds$

$= \cancel{s^2+\frac{s^4}{2}+\cdots+\frac{s^{2n}}{n!}}\quad t^2+\frac{t^4}{2}+\cdots+\frac{t^{2n}}{n!}$

Note that $e^{t^2} = \sum_{i=0}^{\infty}\frac{(t^2)^i}{i!} = 1+t^2+\frac{t^4}{2!}+\cdots$

So $y_n \to e^{t^2}-1$ as $n\to\infty$.

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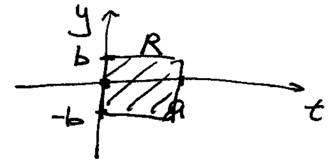
13. $y' = (4y+e^{-t^2})e^{2y}$, $y(0)=0$. $0\le t\le \underbrace{\frac{1}{8\sqrt{e}}}_{=a}$

Let R be the rectangle of $0\le t\le a = \frac{1}{8\sqrt{e}}$, $|y|\le b$

Then $M = \max_{(t,y)\in R}(4y+e^{-t^2})e^{2y} = \max_{|y|\le b}(4y+1)e^{2y}$



$= \cancel{4y+}\ (4b+1)e^{2b}$

Then the solution exists on $0\le t\le \alpha = \min(a, \frac{b}{M}) = \min\left(\frac{1}{8\sqrt{e}}, \frac{b}{(4b+1)e^{2b}}\right)$

So as long as $\frac{b}{(4b+1)e^{2b}} \ge \frac{1}{8\sqrt{e}}$, $\alpha = \frac{1}{8\sqrt{e}}$ and the solution exists on $0\le t\le\frac{1}{8\sqrt{e}}$.

So I want $b \le \frac{1}{4}$

1.10. 2. $y' = t^2 + y^2$. $y(0) = 1$ ($= y_0$)

$$y(t) = 1 + \int_0^t (s^2 + y^2(s))\, ds$$

$$y_1 = 1 + \int_0^t s^2 + 1 = 1 + \frac{t^3}{3} + t$$

$$y_2 = 1 + \int_0^t s^2 + y_1^2(s) = 1 + \int_0^t s^2 + \left(1 + \frac{s^3}{3} + s\right)^2 ds$$

$$= 1 + \int_0^t s^2 + s^2 + 2s + 1 + \frac{s^6}{9} + 2(s+1)\frac{s^3}{3}$$

$$= 1 + \int_0^t \frac{s^6}{9} + \frac{2s^4}{3} + \frac{2}{3}s^3 + 2s^2 + 2s + 1$$

$$= 1 + \frac{t^7}{63} + \frac{2t^5}{15} + \frac{t^4}{6} + \frac{2t^3}{3} + t^2 + t$$

5. $y' = 1 + y + y^2 \cos t$. $y(0) = 0$. $0 \le t \le \frac{1}{3}$

$a = \frac{1}{3}$.

Let R be the rectangle $0 \le t \le \frac{1}{3}$, $|y| \le b$

$$M = \max_{(t,y)\in R} |1 + y + y^2 \cos t| = \max_{|y|\le b} |1 + y + y^2| = 1 + b + b^2$$

So $y(t)$ exists for $0 \le t \le \min\left(\frac{1}{3}, \frac{b}{M}\right) = \min\left(\frac{1}{3}, \frac{b}{1+b+b^2}\right)$

If $\frac{b}{1+b+b^2} \ge \frac{1}{3}$, then $y(t)$ exists for $0 \le t \le \min\left(\frac{1}{3}, \frac{b}{1+b+b^2}\right) = \frac{1}{3}$

w/ $|y| \le b$.

If $\frac{b}{1+b+b^2} \ge \frac{1}{3}$, $b \ge \frac{1}{3} + \frac{1}{3}b + \frac{1}{3}b^2$, $b^2 - 2b + 1 \le 0 \Rightarrow b \le 1$.