Section 1.10

2. $y' = t^2 + y^2$, $y(0) = 1$, compute first two Picard Iterations.

Solution.

$$\frac{dy}{dt} = t^2 + y^2$$

$$y(t) = y(0) + \int_0^t s^2 + y(s)^2 \, ds$$

$$= 1 + \int_0^t s^2 + y(s)^2 \, ds$$

The Picard iteration formula is:

$$y_{n+1}(t) = 1 + \int_0^t s^2 + y_n(s)^2 \, ds, \quad n = 0, 1, 2, \ldots$$

Start from $y_0(t) = 1$. Then

$$y_1(t) = 1 + \int_0^t s^2 + 1^2 \, ds$$

$$= 1 + \frac{t^3}{3} + t$$

$$y_2(t) = 1 + \int_0^t s^2 + y_1(s)^2 \, ds$$

$$= 1 + \int_0^t s^2 + \left(1 + \frac{s^3}{3} + s\right)^2 ds$$

$$= 1 + \int_0^t s^2 + \left[\left(1 + \frac{s^3}{3}\right)^2 + s^2 + 2\left(1 + \frac{s^3}{3}\right) \cdot s\right] ds$$

$$= 1 + \int_0^t s^2 + 1 + \frac{s^6}{9} + \frac{2s^3}{3} + s^2 + 2s + \frac{2s^4}{3} \, ds$$

$$= 1 + \int_0^t 1 + \frac{s^6}{9} + \frac{2s^3}{3} + 2s^2 + 2s + \frac{2s^4}{3} \, ds$$

$$= 1 + t + \frac{t^7}{63} + \frac{2t^4}{12} + \frac{2}{3}t^3 + t^2 + \frac{2t^5}{15}$$

$$= 1 + t + t^2 + \frac{2}{3}t^3 + \frac{1}{6}t^4 + \frac{2}{15}t^5 + \frac{1}{63}t^7.$$

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