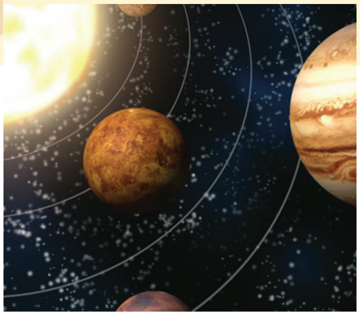
13 Vector Functions



13.2 Derivatives and Integrals of Vector Functions

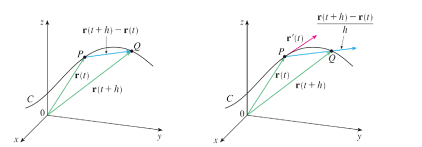
Derivatives

Derivatives

The **derivative r′** of a vector function **r** is defined in much the same way as for real valued functions:

1 $$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector r′(t)

Figure 1

Derivatives

If the points *P* and *Q* have position vectors **r**(*t*) and **r**(*t* + *h*), then $\overrightarrow{PQ}$ represents the vector **r**(*t* + *h*) – **r**(*t*), which can therefore be regarded as a secant vector.

If *h* > 0, the scalar multiple (1/*h*)(**r**(*t* + *h*) – **r**(*t*)) has the same direction as **r**(*t* + *h*) – **r**(*t*). As *h* → 0, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector **r**′(*t*) is called the **tangent vector** to the curve defined by **r** at the point *P*, provided that **r**′(*t*) exists and **r**′(*t*) ≠ **0**.

Derivatives

The **tangent line** to *C* at *P* is defined to be the line through *P* parallel to the tangent vector **r**′(*t*).

We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Derivatives

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of **r**:

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

Example 1 – *Solution*

cont'd

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point (1, 0, 0) is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}$$

$$= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}}$$

$$= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

Derivatives

Just as for real-valued functions, the **second derivative** of a vector function **r** is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function, $\mathbf{r}(t) = \langle 2\cos t, \sin t, t \rangle$, is

$$\mathbf{r}''t = \langle -2\cos t, -\sin t, 0 \rangle$$

Differentiation Rules

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Example 4

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Solution:
Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Example 4 – *Solution*

cont'd

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$.

Integrals

Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions f, g, and h as follows.

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

Integrals

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt \right) \mathbf{i} + \left(\int_a^b g(t)\,dt \right) \mathbf{j} + \left(\int_a^b h(t)\,dt \right) \mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

Example 5

If $\mathbf{r}(t) = 2 \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + 2t\, \mathbf{k}$, then

$$\int \mathbf{r}(t)\, dt = \left(\int 2 \cos t\, dt \right) \mathbf{i} + \left(\int \sin t\, dt \right) \mathbf{j} + \left(\int 2t\, dt \right) \mathbf{k}$$

$$= 2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t)\, dt = \left[2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} \right]_0^{\pi/2}$$

$$= 2\, \mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\, \mathbf{k}$$