12 Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

Three-Dimensional Coordinate Systems

To locate a point in a plane, two numbers are necessary.

We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate.

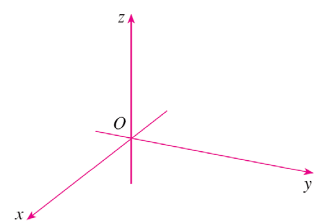
For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required.

We represent any point in space by an ordered triple (a, b, c) of real numbers.

Three-Dimensional Coordinate Systems

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis.

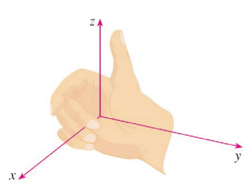
Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1.



Coordinate axes
Figure 1

Three-Dimensional Coordinate Systems

The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2:



Right-hand rule
Figure 2

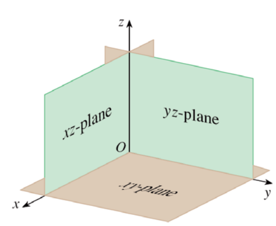
If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

Three-Dimensional Coordinate Systems

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a).

The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes.

These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.



Coordinate planes
Figure 3(a)

Three-Dimensional Coordinate Systems

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)].

Look at any bottom corner of a room and call the corner the origin.

The wall on your left is in the *xz*-plane, the wall on your right is in the *yz*-plane, and the floor is in the *xy*-plane.

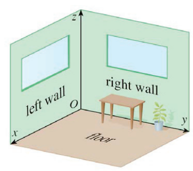


Figure 3(b)

Three-Dimensional Coordinate Systems

The *x*-axis runs along the intersection of the floor and the left wall.

The *y*-axis runs along the intersection of the floor and the right wall.

The *z*-axis runs up from the floor toward the ceiling along the intersection of the two walls.

You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point *O*.

Three-Dimensional Coordinate Systems

Now if *P* is any point in space, let *a* be the (directed) distance from the *yz*-plane to *P*, let *b* be the distance from the *xz*-plane to *P*, and let *c* be the distance from the *xy*-plane to *P*.

We represent the point *P* by the ordered triple (*a*, *b*, *c*) of real numbers and we call *a*, *b*, and *c* the **coordinates** of *P*; *a* is the *x*-coordinate, *b* is the *y*-coordinate, and *c* is the *z*-coordinate.

Three-Dimensional Coordinate Systems

Thus, to locate the point (*a*, *b*, *c*), we can start at the origin *O* and move *a* units along the *x*-axis, then *b* units parallel to the *y*-axis, and then *c* units parallel to the *z*-axis as in Figure 4.

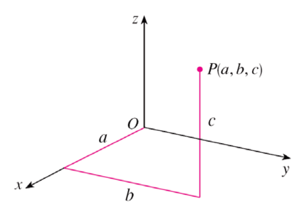


Figure 4

Three-Dimensional Coordinate Systems

The point *P*(*a*, *b*, c) determines a rectangular box as in Figure 5.

If we drop a perpendicular from *P* to the *xy*-plane, we get a point *Q* with coordinates (*a*, *b*, 0) called the **projection** of *P* onto the *xy*-plane.

Similarly, *R*(0, *b*, *c*) and *S*(*a*, 0, *c*) are the projections of *P* onto the *yz*-plane and *xz*-plane, respectively.

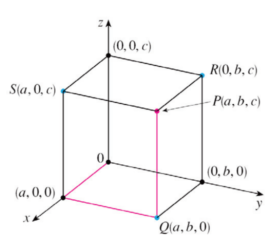


Figure 5

Three-Dimensional Coordinate Systems

As numerical illustrations, the points (–4, 3, –5) and (3, –2, –6) are plotted in Figure 6.

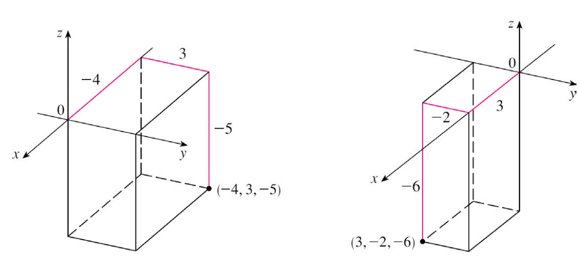


Figure 6

Three-Dimensional Coordinate Systems

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**.

Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Three-Dimensional Coordinate Systems

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$.

In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

Example 1

What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$ **(b)** $y = 5$

Solution:

(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3.

Example 1 – *Solution*

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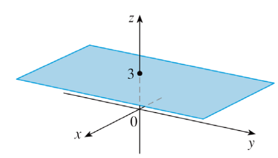
This is the horizontal plane that is parallel to the xy–plane and three units above it as in Figure 7(a).



$z = 3$, a plane in $\mathbb{R}^3$

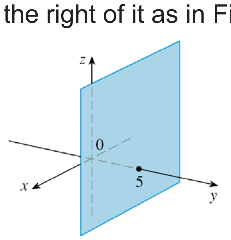
Figure 7(a)

Example 1 – *Solution*

cont'd

(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5.

This is the vertical plane that is parallel to the xz-plane and five units to the right of it as in Figure 7(b).



$y = 5$, a plane in $\mathbb{R}^3$

Figure 7(b)

Three-Dimensional Coordinate Systems

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel to the xy-plane.

In Figure 5, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $z = 0$ (the xy-plane), and the planes $x = a$, $y = b$, and $z = c$.

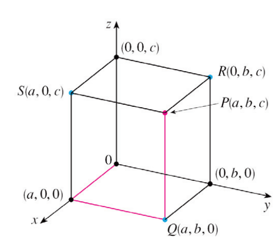


Figure 5

Three-Dimensional Coordinate Systems

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

> **Distance Formula in Three Dimensions** The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is
>
> $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 5

Find an equation of a sphere with radius *r* and center *C*(*h*, *k*, *l*).

Solution:

By definition, a sphere is the set of all points *P*(*x*, *y*, *z*) whose distance from *C* is *r*. (See Figure 12.)

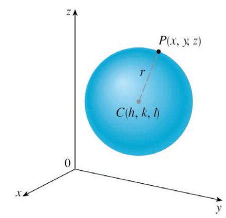


Figure 12

Example 5 – *Solution*

cont'd

Thus *P* is on the sphere if and only if $|PC| = r$.

Squaring both sides, we have

$$|PC|^2 = r^2$$

or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Three-Dimensional Coordinate Systems

The result of Example 5 is worth remembering.

> **Equation of a Sphere** An equation of a sphere with center $C(h, k, l)$ and radius r is
>
> $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$
>
> In particular, if the center is the origin O, then an equation of the sphere is
>
> $$x^2 + y^2 + z^2 = r^2$$