

## Areas and Lengths in Polar Coordinates

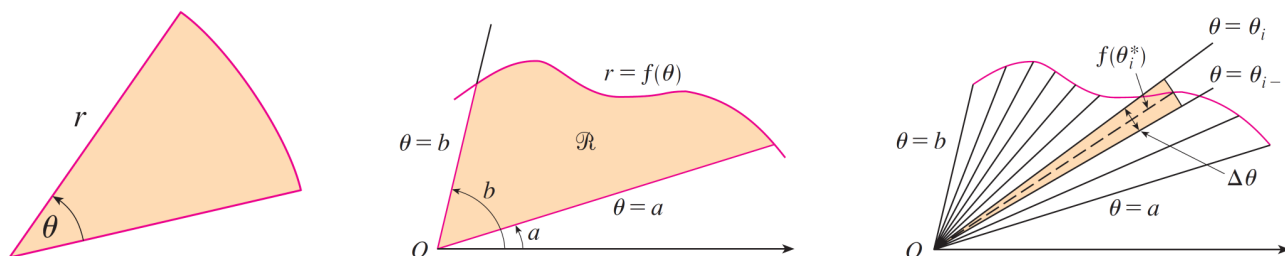
In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle

$$\boxed{A = \frac{1}{2}r^2\theta} \quad (1)$$

where  $r$  is the radius and  $\theta$  is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$$

Let  $\mathcal{R}$  be the region bounded by the polar curve  $r = f(\theta)$  and by the rays  $\theta = a$  and  $\theta = b$ , where  $f$  is a positive continuous function and where  $0 < b - a \leq 2\pi$ .



We divide the interval  $[a, b]$  into subintervals with endpoints  $\theta_0, \theta_1, \theta_2, \dots, \theta_n$  and equal width  $\Delta\theta$ . The rays  $\theta = \theta_i$  then divide  $\mathcal{R}$  into  $n$  smaller regions with central angle  $\Delta\theta = \theta_i - \theta_{i-1}$ . If we choose  $\theta_i^*$  in the  $i$ th subinterval  $[\theta_{i-1}, \theta_i]$ , then the area  $\Delta A_i$  of the  $i$ th region is approximated by the area of the sector of a circle with central angle  $\Delta\theta$  and radius  $f(\theta_i^*)$ . Thus from Formula 1 we have

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta \quad (2)$$

and so an approximation to the total area  $A$  of  $\mathcal{R}$  is  $A \approx \sum_{i=1}^n \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$ . One can see that the approxima-

tion in (2) improves as  $n \rightarrow \infty$ . But the sums in (2) are Riemann sums for the function  $g(\theta) = \frac{1}{2}[f(\theta)]^2$ , so

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

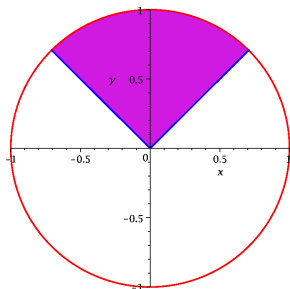
It therefore appears plausible (and can in fact be proved) that the formula for the area  $A$  of the polar region  $\mathcal{R}$  is

$$\boxed{A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta} \quad (3)$$

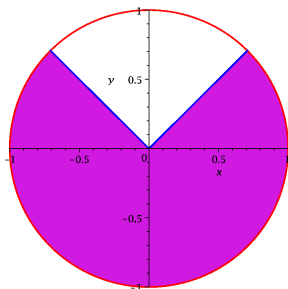
This formula is often written as

$$\boxed{A = \int_a^b \frac{1}{2}r^2 d\theta} \quad (4)$$

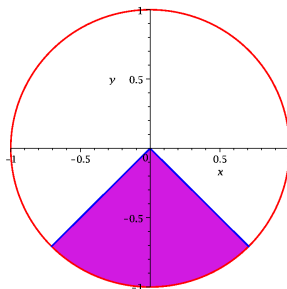
EXAMPLE: Find the area of each of the following regions:



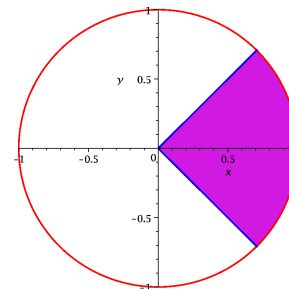
(a)



(b)



(c)



(d)

Solution:

(a) We have

$$A = \int_{\pi/4}^{3\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{\pi/4}^{3\pi/4} d\theta = \frac{1}{2} \left( \frac{3\pi}{4} - \frac{\pi}{4} \right) = \frac{1}{2} \left( \frac{2\pi}{4} \right) = \frac{\pi}{4}$$

(b) We have

$$A = \int_{3\pi/4}^{2\pi+\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{3\pi/4}^{2\pi+\pi/4} d\theta = \frac{1}{2} \left( 2\pi + \frac{\pi}{4} - \frac{3\pi}{4} \right) = \frac{1}{2} \left( 2\pi - \frac{2\pi}{4} \right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

(c) We have

$$A = \int_{5\pi/4}^{7\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{5\pi/4}^{7\pi/4} d\theta = \frac{1}{2} \left( \frac{7\pi}{4} - \frac{5\pi}{4} \right) = \frac{1}{2} \left( \frac{2\pi}{4} \right) = \frac{\pi}{4}$$

(d) We have

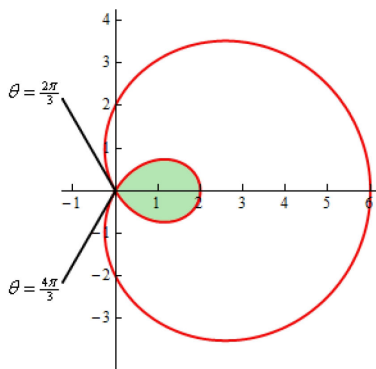
$$A = \int_{7\pi/4}^{2\pi+\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{7\pi/4}^{2\pi+\pi/4} d\theta = \frac{1}{2} \left( 2\pi + \frac{\pi}{4} - \frac{7\pi}{4} \right) = \frac{1}{2} \left( 2\pi - \frac{6\pi}{4} \right) = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

or

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} d\theta = \frac{1}{2} \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = \frac{1}{2} \left( \frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{\pi}{4}$$

EXAMPLE: Find the area of the inner loop of  $r = 2 + 4 \cos \theta$ .

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Solution: We first find  $a$  and  $b$ :

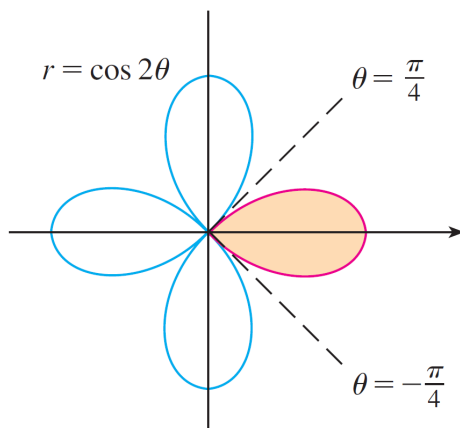
$$2 + 4 \cos \theta = 0 \implies \cos \theta = -\frac{1}{2} \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Therefore the area is

$$\begin{aligned} A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2 + 4 \cos \theta)^2 d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(4 + 16 \cos \theta + 16 \cos^2 \theta) d\theta \\ &= \int_{2\pi/3}^{4\pi/3} (2 + 8 \cos \theta + 8 \cos^2 \theta) d\theta = \int_{2\pi/3}^{4\pi/3} \left( 2 + 8 \cos \theta + 8 \cdot \frac{1 + \cos 2\theta}{2} \right) d\theta \\ &= \int_{2\pi/3}^{4\pi/3} (2 + 8 \cos \theta + 4(1 + \cos 2\theta)) d\theta = \int_{2\pi/3}^{4\pi/3} (6 + 8 \cos \theta + 4 \cos 2\theta) d\theta \\ &= \left[ 6\theta + 8 \sin \theta + 2 \sin 2\theta \right]_{2\pi/3}^{4\pi/3} = 4\pi - 6\sqrt{3} \approx 2.174 \end{aligned}$$

EXAMPLE: Find the area enclosed by one loop of the four-leaved rose  $r = \cos 2\theta$ .

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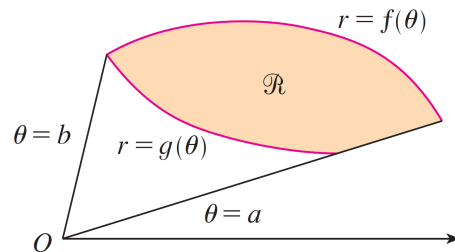


Solution: Notice that the region enclosed by the right loop is swept out by a ray that rotates from  $\theta = -\pi/4$  to  $\theta = \pi/4$ . Therefore, Formula 4 gives

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta = \int_0^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) d\theta = \frac{1}{2} \left[ \theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

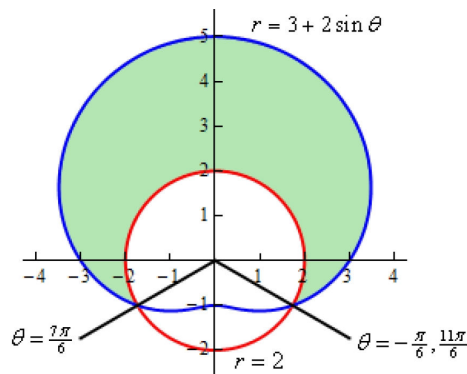
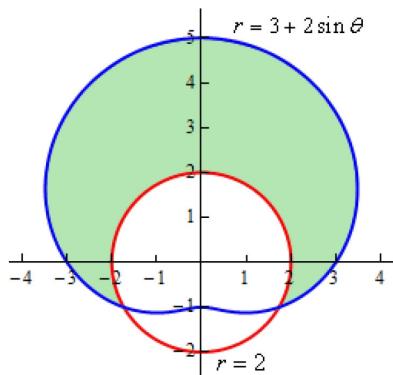
Let  $\mathcal{R}$  be the region bounded by curves with polar equations  $r = f(\theta)$ ,  $r = g(\theta)$ ,  $\theta = a$ , and  $\theta = b$ , where  $f(\theta) \geq g(\theta) \geq 0$  and  $0 < b - a \leq 2\pi$ . Then the area  $A$  of  $\mathcal{R}$  is

$$A = \int_a^b \frac{1}{2} ([f(\theta)]^2 - [g(\theta)]^2) d\theta$$



EXAMPLE: Find the area that lies inside  $r = 3 + 2 \sin \theta$  and outside  $r = 2$ .

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Solution: We first find  $a$  and  $b$ :

$$3 + 2 \sin \theta = 2 \implies \sin \theta = -\frac{1}{2} \implies \theta = \frac{7\pi}{6}, -\frac{\pi}{6} \left( \frac{11\pi}{6} \right)$$

Therefore the area is

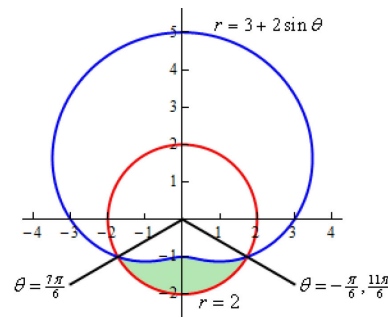
$$\begin{aligned} A &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} [(3 + 2 \sin \theta)^2 - 2^2] d\theta = \int_{-\pi/6}^{7\pi/6} \frac{1}{2} (5 + 12 \sin \theta + 4 \sin^2 \theta) d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} \left( 5 + 12 \sin \theta + 4 \cdot \frac{1 - \cos 2\theta}{2} \right) d\theta = \int_{-\pi/6}^{7\pi/6} \frac{1}{2} (5 + 12 \sin \theta + 2(1 - \cos 2\theta)) d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} (7 + 12 \sin \theta - 2 \cos 2\theta) d\theta = \frac{1}{2} [7\theta - 12 \cos \theta - \sin 2\theta]_{-\pi/6}^{7\pi/6} \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} \approx 24.187 \end{aligned}$$

EXAMPLE: Find the area of the region outside  $r = 3 + 2 \sin \theta$  and inside  $r = 2$ .

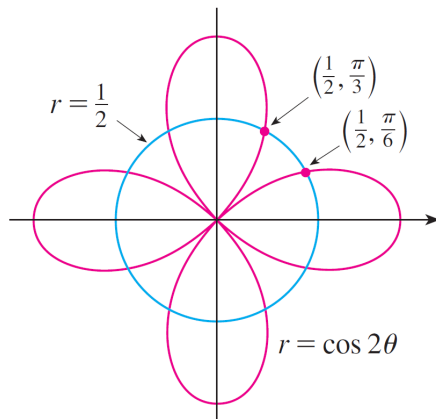
EXAMPLE: Find the area of the region outside  $r = 3 + 2 \sin \theta$  and inside  $r = 2$ .

Solution: We have

$$\begin{aligned} A &= \int_{7\pi/6}^{11\pi/6} \frac{1}{2} [2^2 - (3 + 2 \sin \theta)^2] d\theta \\ &= \int_{7\pi/6}^{11\pi/6} \frac{1}{2} (-5 - 12 \sin \theta - 4 \sin^2 \theta) d\theta \\ &= \int_{7\pi/6}^{11\pi/6} \frac{1}{2} (-7 - 12 \sin \theta + 2 \cos 2\theta) d\theta = \frac{1}{2} [-7\theta + 12 \cos \theta + \sin 2\theta]_{7\pi/6}^{11\pi/6} = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} \approx 2.196 \end{aligned}$$



EXAMPLE: Find all points of intersection of the curves  $r = \cos 2\theta$  and  $r = \frac{1}{2}$ .



Solution: If we solve the equations  $r = \cos 2\theta$  and  $r = \frac{1}{2}$ , we get  $\cos 2\theta = \frac{1}{2}$  and, therefore,

$$2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$$

Thus the values of  $\theta$  between 0 and  $2\pi$  that satisfy both equations are

$$\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$$

We have found four points of intersection:

$$\left(\frac{1}{2}, \pi/6\right), \left(\frac{1}{2}, 5\pi/6\right), \left(\frac{1}{2}, 7\pi/6\right), \text{ and } \left(\frac{1}{2}, 11\pi/6\right)$$

However, you can see from the above figure that the curves have four other points of intersection — namely,

$$\left(\frac{1}{2}, \pi/3\right), \left(\frac{1}{2}, 2\pi/3\right), \left(\frac{1}{2}, 4\pi/3\right), \text{ and } \left(\frac{1}{2}, 5\pi/3\right)$$

These can be found using symmetry or by noticing that another equation of the circle is  $r = -\frac{1}{2}$  and then solving the equations  $r = \cos 2\theta$  and  $r = -\frac{1}{2}$ .

## Arc Length

To find the length of a polar curve  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , we regard  $\theta$  as a parameter and write the parametric equations of the curve as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Using the Product Rule and differentiating with respect to  $\theta$ , we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

So, using  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta = \left(\frac{dr}{d\theta}\right)^2 + r^2 \end{aligned}$$

Assuming that  $f'$  is continuous, we can use one of the formulas from Section 9.2 to write the arc length as

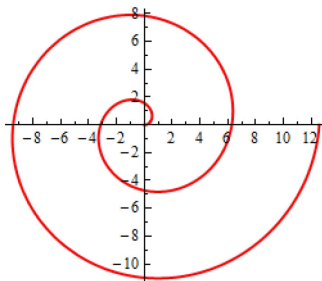
$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Therefore, the length of a curve with polar equation  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , is

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta} \qquad (5)$$

EXAMPLE: Find the length of the curve  $r = \theta$ ,  $0 \leq \theta \leq 1$ .

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Solution: We have

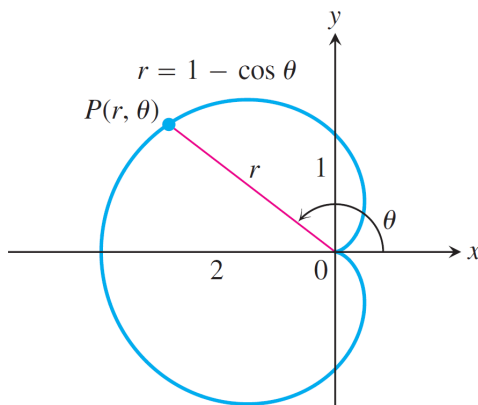
$$L = \int_0^1 \sqrt{\theta^2 + 1} d\theta = \left[ \begin{array}{l} \theta = \tan x \implies \sqrt{\theta^2 + 1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = |\sec x| = \sec x \\ d\theta = d \tan x \\ d\theta = \sec^2 x dx \end{array} \right]$$

$$= \int_0^{\pi/4} \sec^3 x dx = \left[ \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) \right]_0^{\pi/4} = \frac{1}{2} (\sqrt{2} + \ln(1 + \sqrt{2}))$$

EXAMPLE: Find the length of the cardioid  $r = 1 - \cos \theta$ .



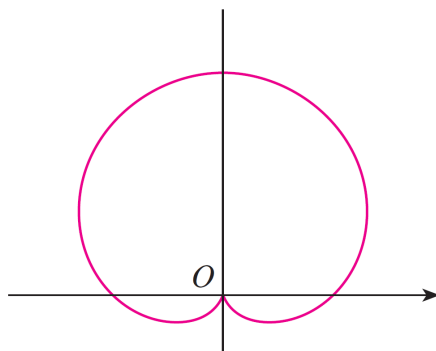
EXAMPLE: Find the length of the cardioid  $r = 1 - \cos \theta$ .



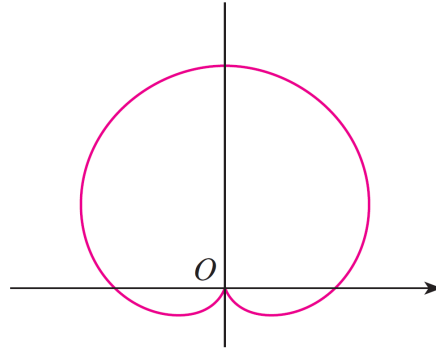
Solution: The full length of the cardioid is given by the parameter interval  $0 \leq \theta \leq 2\pi$ , so Formula 5 gives

$$\begin{aligned}
 L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta \\
 &= \int_0^{2\pi} \sqrt{1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta \\
 &= \int_0^{2\pi} \sqrt{2 - 2\cos \theta} d\theta \\
 &= \int_0^{2\pi} \sqrt{4 \sin^2 \frac{\theta}{2}} d\theta \\
 &= \int_0^{2\pi} 2 \left| \sin \frac{\theta}{2} \right| d\theta \\
 &= \int_0^{2\pi} 2 \sin \frac{\theta}{2} d\theta \\
 &= -4 \cos \frac{\theta}{2} \Big|_0^{2\pi} = 4 + 4 = 8
 \end{aligned}$$

EXAMPLE: Find the length of the cardioid  $r = 1 + \sin \theta$ .



EXAMPLE: Find the length of the cardioid  $r = 1 + \sin \theta$ .



Solution 1: Note that

$$r = 1 + \sin \theta = 1 - \cos \left( \theta + \frac{\pi}{2} \right)$$

Therefore the graph of  $r = 1 + \sin \theta$  is the rotation of the graph of  $r = 1 - \cos \theta$ . Hence the length of the cardioid  $r = 1 + \sin \theta$  is 8 by the previous Example.

Solution 2: The full length of the cardioid is given by the parameter interval  $0 \leq \theta \leq 2\pi$ , so Formula 5 gives

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{2 - 2 \cos \left( \theta + \frac{\pi}{2} \right)} d\theta = \left[ \begin{array}{l} \theta + \frac{\pi}{2} = u \\ d \left( \theta + \frac{\pi}{2} \right) = du \\ d\theta = du \end{array} \right] = \int_{\pi/2}^{5\pi/2} \sqrt{2 - 2 \cos u} du \\ &= \int_{\pi/2}^{5\pi/2} \sqrt{4 \sin^2 \frac{u}{2}} du = \int_{\pi/2}^{5\pi/2} 2 \left| \sin \frac{u}{2} \right| du = \int_{\pi/2}^{2\pi} 2 \left| \sin \frac{u}{2} \right| du + \int_{2\pi}^{5\pi/2} 2 \left| \sin \frac{u}{2} \right| du \\ &= \int_{\pi/2}^{2\pi} 2 \sin \frac{u}{2} du - \int_{2\pi}^{5\pi/2} 2 \sin \frac{u}{2} du \\ &= -4 \cos \frac{u}{2} \Big|_{\pi/2}^{2\pi} + 4 \cos \frac{u}{2} \Big|_{2\pi}^{5\pi/2} \\ &= \left( -4 \cos \pi + 4 \cos \frac{\pi}{4} \right) + \left( 4 \cos \frac{5\pi}{4} - 4 \cos \pi \right) \\ &= (4 + 2\sqrt{2}) + (-2\sqrt{2} + 4) = 8 \end{aligned}$$

Solution 3: The full length of the cardioid is given by the parameter interval  $0 \leq \theta \leq 2\pi$ , so Formula 5 gives

$$\begin{aligned}
 L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} d\theta = \int_0^{2\pi} \sqrt{1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta \\
 &= \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} d\theta = \sqrt{2} \int_0^{2\pi} \sqrt{1 + \sin \theta} d\theta = \sqrt{2} \int_0^{2\pi} \frac{\sqrt{1 + \sin \theta} \sqrt{1 - \sin \theta}}{\sqrt{1 - \sin \theta}} d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \frac{\sqrt{1 - \sin^2 \theta}}{\sqrt{1 - \sin \theta}} d\theta = \sqrt{2} \int_0^{2\pi} \frac{\sqrt{\cos^2 \theta}}{\sqrt{1 - \sin \theta}} d\theta = \sqrt{2} \int_0^{2\pi} \frac{|\cos \theta|}{\sqrt{1 - \sin \theta}} d\theta \\
 &= \sqrt{2} \int_0^{\pi/2} \frac{\cos \theta}{\sqrt{1 - \sin \theta}} d\theta - \sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{\cos \theta}{\sqrt{1 - \sin \theta}} d\theta + \sqrt{2} \int_{3\pi/2}^{2\pi} \frac{\cos \theta}{\sqrt{1 - \sin \theta}} d\theta
 \end{aligned}$$

Note that

$$\begin{aligned}
 \int \frac{\cos \theta}{\sqrt{1 - \sin \theta}} d\theta &= \left[ \begin{array}{l} 1 - \sin \theta = u \\ d(1 - \sin \theta) = du \\ -\cos \theta d\theta = du \\ \cos \theta d\theta = -du \end{array} \right] = - \int \frac{du}{\sqrt{u}} = - \int u^{-1/2} du = -\frac{u^{-1/2+1}}{-1/2+1} + C \\
 &= -2\sqrt{u} + C \\
 &= -2\sqrt{1 - \sin \theta} + C
 \end{aligned}$$

Therefore

$$\begin{aligned}
 L &= -2\sqrt{2}\sqrt{1 - \sin \theta} \Big|_0^{\pi/2} + 2\sqrt{2}\sqrt{1 - \sin \theta} \Big|_{\pi/2}^{3\pi/2} - 2\sqrt{2}\sqrt{1 - \sin \theta} \Big|_{3\pi/2}^{2\pi} \\
 &= -2\sqrt{2} \left( \sqrt{1 - \sin(\pi/2)} - \sqrt{1 - \sin 0} \right) \\
 &\quad + 2\sqrt{2} \left( \sqrt{1 - \sin(3\pi/2)} - \sqrt{1 - \sin(\pi/2)} \right) \\
 &\quad - 2\sqrt{2} \left( \sqrt{1 - \sin(2\pi)} - \sqrt{1 - \sin(3\pi/2)} \right) \\
 &= -2\sqrt{2}(0 - 1) + 2\sqrt{2}(\sqrt{2} - 0) - 2\sqrt{2}(1 - \sqrt{2}) \\
 &= 2\sqrt{2} + 4 - 2\sqrt{2} + 4 = 8
 \end{aligned}$$