Orthogonal Diagonalization

Definition Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. A linear operator $L: \mathcal{V} \to \mathcal{V}$ is a **symmetric operator** on $\mathcal{V}$ if and only if $L(\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot L(\mathbf{v}_2)$, for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$.

Example 1

The operator L on $\mathbb{R}^3$ given by $L([a,b,c]) = [b,a,-c]$ is symmetric since

$$L([a,b,c]) \cdot [d,e,f] = [b,a,-c] \cdot [d,e,f] = bd + ae - cf$$

$$\text{and} \quad [a,b,c] \cdot L([d,e,f]) = [a,b,c] \cdot [e,d,-f] = ae + bd - cf.$$

You can verify that the matrix representation for the operator L in Example 1 with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

a symmetric matrix. The next theorem asserts that an operator on a subspace $\mathcal{V}$ of $\mathbb{R}^n$ is symmetric if and only if its matrix representation with respect to any orthonormal basis for $\mathcal{V}$ is symmetric.

Theorem 6.18 Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, L be a linear operator on $\mathcal{V}$, B be an ordered orthonormal basis for $\mathcal{V}$, and $\mathbf{A}$ be the matrix for L with respect to B. Then L is a symmetric operator if and only if $\mathbf{A}$ is a symmetric matrix.

Lemma 6.19 Let L be a symmetric operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$. Then L has at least one eigenvalue.

Definition Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, and let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator. Then L is an **orthogonally diagonalizable operator** if and only if there is an ordered orthonormal basis B for $\mathcal{V}$ such that the matrix for L with respect to B is a diagonal matrix.

A square matrix $\mathbf{A}$ is **orthogonally diagonalizable** if and only if there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

Theorem 6.20 Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, and let L be a linear operator on $\mathcal{V}$. Then L is orthogonally diagonalizable if and only if L is symmetric.

Method for Orthogonally Diagonalizing a Symmetric Operator (Orthogonal Diagonalization Method)

Let $L: \mathcal{V} \to \mathcal{V}$ be a symmetric operator on a subspace $\mathcal{V}$ of $\mathbb{R}^n$, with $\dim(\mathcal{V}) = k$.

Step 1: Find an ordered orthonormal basis C for $\mathcal{V}$ (if $\mathcal{V} = \mathbb{R}^n$, we can use the standard basis), and calculate the matrix representation $\mathbf{A}$ for L with respect to C (which should be a $k \times k$ symmetric matrix).

Step 2: (a) Apply the Diagonalization Method of Section 3.4 to $\mathbf{A}$ in order to obtain all of the eigenvalues $\lambda_1, \ldots, \lambda_m$ of $\mathbf{A}$, and a basis in $\mathbb{R}^k$ for each eigenspace E_{λ_i} of $\mathbf{A}$ (by solving an appropriate homogeneous system if necessary).

(b) Perform the Gram-Schmidt Process on the basis for each E_{λ_i} from Step 2(a), and then normalize to get an orthonormal basis for each E_{λ_i}.

(c) Let $Z = (\mathbf{z}_1 \ldots, \mathbf{z}_k)$ be an ordered basis for $\mathbb{R}^k$ consisting of the union of the orthonormal bases for the E_{λ_i}.

Step 3: Reverse the C-coordinatization isomorphism on the vectors in Z to obtain an ordered orthonormal basis $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ for $\mathcal{V}$; that is, $[\mathbf{v}_i]_C = \mathbf{z}_i$.

The matrix representation for L with respect to B is the diagonal matrix $\mathbf{D}$, where d_{ii} is the eigenvalue for L corresponding to $\mathbf{v}_i$. In most practical situations, the transition matrix $\mathbf{P}$ from B- to C-coordinates is useful. $\mathbf{P}$ is the $k \times k$ matrix whose columns are $[\mathbf{v}_1]_C, \ldots, [\mathbf{v}_k]_C$ — that is, the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_k$ in Z. Note that $\mathbf{P}$ is an orthogonal matrix, and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P}$.

Example 3

Consider the operator $L: \mathbb{R}^4 \to \mathbb{R}^4$ given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$, where

$$\mathbf{A} = \frac{1}{7}\begin{bmatrix} 15 & -21 & -3 & -5 \\ -21 & 35 & -7 & 0 \\ -3 & -7 & 23 & 15 \\ -5 & 0 & 15 & 39 \end{bmatrix}.$$

L is clearly symmetric, since its matrix $\mathbf{A}$ with respect to the standard basis C for $\mathbb{R}^4$ is symmetric. We find an orthonormal basis B such that the matrix for L with respect to B is diagonal.

Step 1: We have already seen that $\mathbf{A}$ is the matrix for L with respect to the standard basis C for $\mathbb{R}^4$.

Step 2: (a) A lengthy calculation yields

$$p_{\mathbf{A}}(x) = x^4 - 16x^3 + 77x^2 - 98x = x(x-2)(x-7)^2,$$

giving eigenvalues $\lambda_1 = 0, \lambda_2 = 2$, and $\lambda_3 = 7$. Solving the appropriate homogeneous systems to find bases for the eigenspaces produces

$$\begin{aligned} \text{Basis for } E_{\lambda_1} &= \{[3,2,1,0]\} \\ \text{Basis for } E_{\lambda_2} &= \{[1,0,-3,2]\} \\ \text{Basis for } E_{\lambda_3} &= \{[-2,3,0,1],[3,-5,1,0]\}. \end{aligned}$$

(b) There is no need to perform the Gram-Schmidt Process on the bases for E_{λ_1} and E_{λ_2}, since each of these eigenspaces is one-dimensional. Normalizing the basis vectors yields

$$\text{Orthonormal basis for } E_{\lambda_1} = \left\{ \frac{1}{\sqrt{14}}[3,2,1,0] \right\}$$

$$\text{Orthonormal basis for } E_{\lambda_2} = \left\{ \frac{1}{\sqrt{14}}[1,0,-3,2] \right\}.$$

Let us label the vectors in these bases as $\mathbf{z}_1, \mathbf{z}_2$, respectively. However, we must perform the Gram-Schmidt Process on the basis for E_{λ_3}. Let $\mathbf{w}_1 = \mathbf{v}_1 = [-2,3,0,1]$ and $\mathbf{w}_2 = [3,-5,1,0]$. Then

$$\mathbf{v}_2 = [3,-5,1,0] - \left(\frac{[3,-5,1,0]\cdot[-2,3,0,1]}{[-2,3,0,1]\cdot[-2,3,0,1]} \right) [-2,3,0,1] = \left[0, -\frac{1}{2}, 1, \frac{3}{2}\right].$$

Finally, normalizing $\mathbf{v}_1$ and $\mathbf{v}_2$, we obtain

$$\text{Orthonormal basis for } E_{\lambda_3} = \left\{ \frac{1}{\sqrt{14}}[-2,3,0,1], \frac{1}{\sqrt{14}}[0,-1,2,3] \right\}.$$

Let us label the vectors in this basis as $\mathbf{z}_3, \mathbf{z}_4$, respectively.

(c) We let $Z = (\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) =$

$$\left(\frac{1}{\sqrt{14}}[3,2,1,0], \frac{1}{\sqrt{14}}[1,0,-3,2], \frac{1}{\sqrt{14}}[-2,3,0,1], \frac{1}{\sqrt{14}}[0,-1,2,3] \right)$$

be the union of the orthonormal bases for $E_{\lambda_1}, E_{\lambda_2}$, and E_{λ_3}.

Step 3: Since C is the standard basis for $\mathbb{R}^4$, the C-coordinatization isomorphism is the identity mapping, so $\mathbf{v}_1 = \mathbf{z}_1$, $\mathbf{v}_2 = \mathbf{z}_2, \mathbf{v}_3 = \mathbf{z}_3$, and $\mathbf{v}_4 = \mathbf{z}_4$ here, and $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is an ordered orthonormal basis for $\mathbb{R}^4$. The matrix representation $\mathbf{D}$ of L with respect to B is

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

The transition matrix $\mathbf{P}$ from B to C is the *orthogonal* matrix

$$\mathbf{P} = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 & 1 & -2 & 0 \\ 2 & 0 & 3 & -1 \\ 1 & -3 & 0 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix}.$$

You can verify that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$.

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